On common generalized inverses of a pair of matrices

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Necessary and sufficient conditions are established for a pair of matrices of the same size to have a common $\{1\}$-inverse, $\{1,2\}$-inverse, $\{1,3\}$-inverse, $\{1,4\}$-inverse, $\{1,2,3\}$-inverse and $\{1,2,4\}$-inverse, $\{1,3,4\}$-inverse, respectively. The relations between $(A^*)^-$ and $(A^-)^*$ are also investigated. In addition, some consequences and applications are given.

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1 Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices over the field of complex numbers. A matrix $X \in \mathbb{C}^{n\times m}$ is called a generalized inverse (g-inverse) or $\{1\}$-inverse of $A \in \mathbb{C}^{m\times n}$, denoted by A^-, if it satisfies $AXA = A$, while the collection of all A^- is denoted by $\{A^-\}$. In addition to A^-, the definitions of some other well-known generalized inverses of A are given as follows: the Moore-Penrose inverse of A, denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four Penrose equations

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA,$$

where $(\cdot)^*$ denotes the conjugate transpose of a complex matrix. Suppose $\{i, \ldots, j\}$ is a nonempty subset of $\{1,\ 2,\ 3,\ 4\}$. An X is called an $\{i, \ldots, j\}$-inverse of A if it satisfies the $i, \ldots, j$th equations and is denoted by $A^{(i,\ldots,j)}$; the collection of all $\{i, \ldots, j\}$-inverses of A is denoted by $\{A^{(i,\ldots,j)}\}$. In particular, a $\{1,\ 2\}$-inverse of A, also denoted by A_r^-, is also called a reflexive g-inverse of A; a $\{1,\ 3\}$-inverse of A is also called a least-squares g-inverse of A; a $\{1,\ 4\}$-inverse of A is also called a minimum norm g-inverse of A. $\{1,2,3\}$-inverse, $\{1,2,4\}$-inverse and $\{1,3,4\}$-inverse of A are defined similarly. The seven g-inverses $A^{(1)}$, $A^{(1,2)}$, $A^{(1,3)}$, $A^{(1,4)}$, $A^{(1,2,3)}$, $A^{(1,2,4)}$ and $A^{(1,3,4)}$ of A have been studied by lots of authors; see, e.g., [1, 2, 6, 14] among others.

Suppose A is a square matrix. A Hermitian matrix X is called a Hermitian $\{i, \ldots, j\}$-inverse of A if it satisfies the $i, \ldots, j$th equations and is denoted by $A_h^{(i,\ldots,j)}$. Hermitian $\{i, \ldots, j\}$-inverse of A if it satisfies the $i, \ldots, j$th equations and is denoted by $A_h^{(i,\ldots,j)}$. In particular, $A_h^{(1)}$, $A_h^{(1,2)}$, $A_h^{(1,3)}$, $A_h^{(1,4)}$, $A_h^{(1,2,3)}$, $A_h^{(1,2,4)}$ and $A_h^{(1,3,4)}$ are seven Hermitian inverses of A. It should be pointed out that the Hermitian $\{i, \ldots, j\}$-inverse of A does not necessarily exist.

Suppose A and B are singular matrices of the same size. Then their g-inverses are not unique. In this case, it is of interest to see whether these two matrices have a common g-inverse. Precisely, one may want to know

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(a) The existence of A^- and B^- so that $A^- = B^-$.

(b) Necessary and sufficient conditions for $\{A^-\} \subseteq \{B^-\}$ and $\{A^-\} = \{B^-\}$ to hold.

(c) Necessary and sufficient conditions for $\{AA^-\} \subseteq \{BB^-\}$, $\{A^-A\} \subseteq \{B^-B\}$, $\{AA^-\} = \{BB^-\}$ and $\{A^-A\} = \{B^-B\}$ to hold.

Mitra [4, 5] showed that $\{A^-\} = \{B^-\}$ if and only if $A = B$. Some other results on common g-inverses of two matrices were derived in Tian [8, 9] by the matrix rank method. In this paper, we seek necessary and sufficient conditions such that A and B of the same size have a common $\{1\}$-inverse, $\{1,2\}$-inverse, $\{1,3\}$-inverse, $\{1,4\}$-inverse, $\{1,2,3\}$-inverse, $\{1,2,4\}$-inverse and $\{1,3,4\}$-inverse, respectively. A variety of consequences and applications are also given, including a group of results on common g-inverses of a square matrix A and its conjugate transpose A^*.

It is well known that the general expressions of g-inverses of A can be written as the following linear matrix expressions

$$A^- = A^\dagger + F_AV + WE_A, \tag{1.1}$$

$$A_r^- = (A^\dagger + F_AV)A(A^\dagger + WE_A), \tag{1.2}$$

$$A^{(1,3)} = A^\dagger + F_AV, \tag{1.3}$$

$$A^{(1,4)} = A^\dagger + WE_A, \tag{1.4}$$

$$A^{(1,2,3)} = A^\dagger + F_AVAA^\dagger, \tag{1.5}$$

$$A^{(1,2,4)} = A^\dagger + A^\dagger AWE_A, \tag{1.6}$$

$$A^{(1,3,4)} = A^\dagger + F_AVE_A, \tag{1.7}$$

where $E_A = I - AA^\dagger$, $F_A = I - A^\dagger A$, the two matrices V and W are arbitrary; see [1, 2]. Various properties of g-inverses can be derived from these matrix expressions.

It is obvious that two matrices A and B of the same size have a common $\{i,\ldots,j\}$-inverse if and only if

$$\min_{A^{(i,\ldots,j)},\, B^{(i,\ldots,j)}} r(A^{(i,\ldots,j)} - B^{(i,\ldots,j)}) = 0,$$

where $r(\cdot)$ denotes the rank of a matrix. If one can establish a formula for the minimal rank on the left-hand side of this equality, necessary and sufficient conditions for $A^{(i,\ldots,j)} = B^{(i,\ldots,j)}$ to hold can be derived from this formula.

In the past several years, one of the authors gave a set of formulas for the extremal ranks of some simple linear matrix expressions through generalized inverses of matrices:

$$\min_X r(A - BXC) = r[A,\, B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{1.8}$$

$$\min_{X,\,Y} r(A - BX - YC) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C), \tag{1.9}$$

where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$, and $[A,\, B]$ denotes a row block matrix. The matrices X and Y satisfying (1.8) and (1.9) can be expressed in generalized inverses, see [7, 8, 12]. A general result is (see [9])

$$\min_{X_1,\,X_2} r(A - B_1X_1C_1 - B_2X_2C_2) = r\begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} + r[A,\, B_1,\, B_2]$$
$$+ \max\left\{ r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}, \right.$$

$$r\begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}\Bigg\}. \tag{1.10}$$

These fundamental formulas can be applied for finding extremal ranks of various matrix expressions that involve variant matrices. For instance, suppose $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times m}$. Then

$$\max_{A^{(1,3)}} r(\, D - CA^{(1,3)}\,) = \min\left\{ m, \;\; r\begin{bmatrix} A^*A & A^* \\ C & D \end{bmatrix} - r(A) \right\}, \tag{1.11}$$

$$\min_{A^{(1,3)}} r(\, D - CA^{(1,3)}\,) = r\begin{bmatrix} A^*A & A^* \\ C & D \end{bmatrix} - r\begin{bmatrix} A \\ C \end{bmatrix}. \tag{1.12}$$

Suppose $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $D \in \mathbb{C}^{n\times k}$. Then

$$\max_{A^{(1,4)}} r(\, D - A^{(1,4)}B\,) = \min\left\{ n, \;\; r\begin{bmatrix} AA^* & B \\ A^* & D \end{bmatrix} - r(A) \right\}, \tag{1.13}$$

$$\min_{A^{(1,4)}} r(\, D - A^{(1,4)}B\,) = r\begin{bmatrix} AA^* & B \\ A^* & D \end{bmatrix} - r[\, A, \, B\,]. \tag{1.14}$$

The proofs of these results are given in [10]. Another simple result on ranks of two matrices is

if $PA = B$ and $QB = A$ for some matrices P and Q, then $r(A) = r(B)$.

It is easy to derive from this result that

$$r\begin{bmatrix} AA^{\dagger} & A \\ BB^{\dagger} & B \end{bmatrix} = r\begin{bmatrix} A^* & A^*A \\ B^* & B^*B \end{bmatrix}. \tag{1.15}$$

This rank equality will be used in the sequel.

2 Main results

The problem on common $\{1\}$-inverses of a pair of matrices A and B of the same size was investigated by one of the authors through some rank formulas. The following result is given in [9].

Theorem 2.1 *Let $A, \; B \in \mathbb{C}^{m\times n}$. Then*

$$\max_{B^-} r(\, A - AB^-A\,) = \min\{\, r(A), \quad r(\, B - A\,) - r(B) + r(A)\,\}, \tag{2.1}$$

$$\min_{B^-} r(\, A - AB^-A\,) = \min_{A^-,\, B^-} r(\, A^- - B^-\,)$$
$$= r(\, A - B\,) + r(A) + r(B) - r[\, A, \, B\,] - r\begin{bmatrix} A \\ B \end{bmatrix}. \tag{2.2}$$

Hence,

(a) *A and B have a common $\{1\}$-inverse if and only if*

$$r(\, A - B\,) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[\, A, \, B\,] - r(A) - r(B). \tag{2.3}$$

(b) *The set inclusion $\{B^-\} \subseteq \{A^-\}$ holds if and only if $r(\, B - A\,) = r(B) - r(A)$.*

(c) $\{A^-\} = \{B^-\}$ *if and only if* $A = B$.

(d) $\{A^-\} \cap \{B^-\} = \{0\}$ *if and only if* $r(A - B) > r\begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] - r(A) - r(B)$.

(e) *If* $\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$ *and* $\mathscr{R}(A^*) \cap \mathscr{R}(B^*) = \{0\}$, *then there exist* A^- *and* B^- *such that* $A^- = B^-$.

Theorem 2.1(b) and (c) were shown in [4]. Theoretically, the existence of common $\{1,2\}$-inverse of a pair of matrices A and B can be determined through the minimal rank of $A_r^- - B_r^-$. Note that the general expression of $A_r^- - B_r^-$ is

$$A_r^- - B_r^- = (A^\dagger + F_A V_1) A (A^\dagger + W_1 E_A) - (B^\dagger + F_B V_2) B (B^\dagger + W_2 E_B).$$

This is a quadratic matrix expression with four variant matrices V_1, V_2, W_1 and W_2. There is, however, no formula available at present for finding the minimal rank of this expression. Instead, it is shown in [11] that if a pair of matrix equations $AXA = A$ and $BXB = B$ have a common solution, then

$$\min_{\substack{AXA = A \\ BXB = B}} r(X) = \max\{ r(A), \ r(B) \}. \tag{2.4}$$

In light of (2.4), we can show the following result.

Theorem 2.2 *Let* $A, B \in \mathbb{C}^{m \times n}$. *Then* A *and* B *have a common* $\{1,2\}$*-inverse if and only if the following two rank equalities*

$$r(A - B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] - r(A) - r(B) \ \text{and} \ r(A) = r(B) \tag{2.5}$$

hold.

Proof Recall that $A^- \in \{A_r^-\}$ if and only if $r(A^-) = r(A)$. Also note that $\{A_r^-\} \subseteq \{A^-\}$. Hence, the two matrices A and B have a common $\{1,2\}$-inverse if and only if $AXA = A$ and $BXB = B$ have a common solution and

$$\min_{\substack{AXA = A \\ BXB = B}} r(X) = r(A) = r(B). \tag{2.6}$$

It can be seen from Theorem 2.1(a) that A and B have a common $\{1\}$-inverse, i.e., $AXA = A$ and $BXB = B$ have a common solution, if and only if (2.3) holds. In this case, the minimal rank of common solutions is given in (2.4). Combining (2.3), (2.4) and (2.6) gives (2.5). □

It was shown in [13] that any two idempotent matrices A and B of the same order satisfy (2.3). Hence, it can be seen from Theorem 2.2 that two idempotent matrices A and B of the same order have a common reflexive g-inverse if and only if $r(A) = r(B)$.

Theorem 2.3 *Let* $A, B \in \mathbb{C}^{m \times n}$. *Then*

$$\min_{A^{(1,3)},\, B^{(1,3)}} r(A^{(1,3)} - B^{(1,3)}) = s, \tag{2.7}$$

$$\min_{A^{(1,3)}} r(BB^\dagger - BA^{(1,3)}) = s, \tag{2.8}$$

$$\min_{B^{(1,3)}} r(AA^\dagger - AB^{(1,3)}) = s, \tag{2.9}$$

where $s = r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix} - r\begin{bmatrix} A \\ B \end{bmatrix}$. *Hence, A and B have a common $\{1,3\}$-inverse if and only if*

$$r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix} = r\begin{bmatrix} A \\ B \end{bmatrix}, \quad i.e., \ \mathscr{R}\begin{bmatrix} A^* \\ B^* \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A^*A \\ B^*B \end{bmatrix}. \tag{2.10}$$

Proof From (1.3), the general expression of $A^{(1,3)} - B^{(1,3)}$ is given by

$$A^{(1,3)} - B^{(1,3)} = A^{\dagger} - B^{\dagger} + F_AV_1 + F_BV_2 = A^{\dagger} - B^{\dagger} + [\, F_A, \, F_B \,]V,$$

where $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$. Hence, by (1.8)

$$\begin{aligned} \min_{A^{(1,3)},\, B^{(1,3)}} r(\, A^{(1,3)} - B^{(1,3)} \,) &= \max_V r(\, A^{\dagger} - B^{\dagger} + [\, F_A, \, F_B \,]V \,) \\ &= r[\, A^{\dagger} - B^{\dagger}, \, F_A, \, F_B \,] - r[\, F_A, \, F_B \,]. \end{aligned} \tag{2.11}$$

Simplifying the ranks of the two block matrices in (2.11) by elementary block matrix operations (these operations do not change the rank of a matrix) and the following rank formulas for partitioned matrices due to Marsaglia and Styan (see [3])

$$r[\, A, \, B \,] = r(A) + r(\, B - AA^{\dagger}B \,), \tag{2.12}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(\, C - CA^{\dagger}A \,), \tag{2.13}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r[\, (I_m - BB^{\dagger})A(I_n - C^{\dagger}C) \,] \tag{2.14}$$

gives us

$$\begin{aligned} r[\, A^{\dagger} - B^{\dagger}, \; F_A, \, F_B \,] &= r\begin{bmatrix} A^{\dagger} - B^{\dagger} & I_n & I_n \\ 0 & A & 0 \\ 0 & 0 & B \end{bmatrix} - r(A) - r(B) \\ &= r\begin{bmatrix} 0 & I_n & 0 \\ -AA^{\dagger} & 0 & -A \\ BB^{\dagger} & 0 & B \end{bmatrix} - r(A) - r(B) \\ &= r\begin{bmatrix} -AA^{\dagger} & -A \\ BB^{\dagger} & B \end{bmatrix} + n - r(A) - r(B) \\ &= r\begin{bmatrix} A^* & A^*A \\ B^* & B^*B \end{bmatrix} + n - r(A) - r(B) \quad \text{(by (1.15))} \\ &= r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix} + n - r(A) - r(B), \\ r[\, F_A, \, F_B \,] &= r\begin{bmatrix} I_n & I_n \\ A & 0 \\ 0 & B \end{bmatrix} - r(A) - r(B) \\ &= r\begin{bmatrix} I_n & 0 \\ 0 & -A \\ 0 & B \end{bmatrix} - r(A) - r(B) \\ &= r\begin{bmatrix} A \\ B \end{bmatrix} + n - r(A) - r(B). \end{aligned}$$

Substituting these results into (2.11) yields (2.7). It is well known that $X \in \{A^{(1,3)}\}$ if and only if $AX = AA^{\dagger}$; see, e.g., [1]. Applying (1.12) to $BB^{\dagger} - BA^{(1,3)}$ and $AA^{\dagger} - AB^{(1,3)}$ gives (2.8) and (2.9). □

Theorem 2.4 *Let $A,\, B \in \mathbb{C}^{m\times n}$. Then*

$$\max_{A^{(1,3)}} r(\, BB^{\dagger} - BA^{(1,3)}\,) = r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix} - r(A), \tag{2.15}$$

$$\max_{B^{(1,3)}} r(\, AA^{\dagger} - AB^{(1,3)}\,) = r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix} - r(B). \tag{2.16}$$

Hence,

(a) $\{A^{(1,3)}\} \subseteq \{B^{(1,3)}\} \Leftrightarrow r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix} = r(A)$, *i.e.*, $\mathscr{R}\begin{bmatrix} B \\ B^*B \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A \\ A^*A \end{bmatrix}$.

(b) $\{B^{(1,3)}\} \subseteq \{A^{(1,3)}\} \Leftrightarrow r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix} = r(B)$, *i.e.*, $\mathscr{R}\begin{bmatrix} A \\ A^*A \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} B \\ B^*B \end{bmatrix}$.

(c) $\{A^{(1,3)}\} = \{B^{(1,3)}\} \Leftrightarrow A = B$.

Proof Recall that $X \in \{A^{(1,3)}\}$ if and only if $AX = AA^{\dagger}$ and the set inclusion $\{A^{(1,3)}\} \subseteq \{B^{(1,3)}\}$ holds if and only if $\max_{A^{(1,3)}} r(\, BB^{\dagger} - BA^{(1,3)}\,) = 0$. Applying (1.11) to $BB^{\dagger} - BA^{(1,3)}$ and $AA^{\dagger} - AB^{(1,3)}$ gives (2.15) and (2.16). It can be seen from (a) and (b) of this theorem that $\{A^{(1,3)}\} = \{B^{(1,3)}\}$ if and only if

$$r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix} = r(A) = r(B),$$

which are equivalent to

$$r\begin{bmatrix} A & B \\ 0 & B^*B - A^*B \end{bmatrix} = r(A) \quad \text{and} \quad r\begin{bmatrix} A & B \\ A^*A - B^*A & 0 \end{bmatrix} = r(B).$$

Hence, $B^*B = A^*B$ and $A^*A = B^*A$ hold. In this case,

$$(\, A - B\,)^*(\, A - B\,) = A^*A - B^*A - A^*B + B^*B = 0,$$

which implies $A = B$. □

The two rank formulas in (2.8) and (2.15) are also mentioned in [10]. The following result can be obtained by a similar approach.

Theorem 2.5 *Let $A,\, B \in \mathbb{C}^{m\times n}$. Then*

$$\min_{A^{(1,4)},\, B^{(1,4)}} r(\, A^{(1,4)} - B^{(1,4)}\,) = s - r[\, A,\, B\,], \tag{2.17}$$

$$\min_{A^{(1,4)}} r(\, B^{\dagger}B - A^{(1,4)}B\,) = s - r[\, A,\, B\,], \tag{2.18}$$

$$\min_{B^{(1,4)}} r(\, A^{\dagger}A - B^{(1,4)}A\,) = s - r[\, A,\, B\,], \tag{2.19}$$

$$\max_{A^{(1,4)}} r(\, B^{\dagger}B - A^{(1,4)}B\,) = s - r(A), \tag{2.20}$$

$$\max_{B^{(1,3)}} r(\, A^{\dagger}A - B^{(1,4)}A\,) = s - r(B), \tag{2.21}$$

where $s = r\begin{bmatrix} A & AA^* \\ B & BB^* \end{bmatrix}$. *Hence,*

(a) *A and B have a common* $\{1,4\}$*-inverse* $\Leftrightarrow r\begin{bmatrix} A & AA^* \\ B & BB^* \end{bmatrix} = r[A, \ B],$ *i.e.,* $\mathscr{R}\begin{bmatrix} A \\ B \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} AA^* \\ BB^* \end{bmatrix}.$

(b) $\{A^{(1,4)}\} \subseteq \{B^{(1,4)}\} \Leftrightarrow r\begin{bmatrix} A & AA^* \\ B & BB^* \end{bmatrix} = r(A),$ *i.e.,* $\mathscr{R}\begin{bmatrix} B^* \\ BB^* \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A^* \\ AA^* \end{bmatrix}.$

(c) $\{B^{(1,4)}\} \subseteq \{A^{(1,4)}\} \Leftrightarrow r\begin{bmatrix} A & AA^* \\ B & BB^* \end{bmatrix} = r(B),$ *i.e.,* $\mathscr{R}\begin{bmatrix} A^* \\ AA^* \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} B^* \\ BB^* \end{bmatrix}.$

(d) $\{A^{(1,4)}\} = \{B^{(1,4)}\} \Leftrightarrow A = B.$

Theorem 2.6 *Let* $A, \, B \in \mathbb{C}^{m \times n}$*. Then*

$$\min_{A^{(1,3)}, \, B^{(1,4)}} r(\, A^{(1,3)} - B^{(1,4)}\,) = r(\, BA^*A - BB^*A\,). \tag{2.22}$$

Hence, there are $A^{(1,3)}$ *and* $B^{(1,4)}$ *such that* $A^{(1,3)} = B^{(1,4)}$ *if and only if* $BA^*A = BB^*A.$

Proof In terms of (1.3) and (1.4), the general expression of $A^{(1,3)} - B^{(1,4)}$ is given by

$$A^{(1,3)} - B^{(1,4)} = A^{\dagger} - B^{\dagger} + F_AV + WE_B,$$

where V and W are arbitrary. Hence by (1.9) and (2.14), we get

$$\begin{aligned} &\min_{A^{(1,3)}, \, B^{(1,4)}} r(\, A^{(1,3)} - B^{(1,4)}\,) \\ &= \min_{V, \, W} r(\, A^{\dagger} - B^{\dagger} + F_AV + WE_B\,) \\ &= r\begin{bmatrix} A^{\dagger} - B^{\dagger} & F_A \\ E_B & 0 \end{bmatrix} - r(F_A) - r(E_B) \\ &= r\begin{bmatrix} A^{\dagger} - B^{\dagger} & I_n & 0 \\ I_m & 0 & B \\ 0 & A & 0 \end{bmatrix} - m - n \\ &= r\begin{bmatrix} 0 & I_n & 0 \\ I_m & 0 & 0 \\ 0 & 0 & AA^{\dagger}B - AB^{\dagger}B \end{bmatrix} - m - n \\ &= r(\, AA^{\dagger}B - AB^{\dagger}B\,). \end{aligned}$$

Also note that

$$A^*(\, AA^{\dagger}B - AB^{\dagger}B\,)B^* = A^*BB^* - A^*AB^*,$$
$$(A^{\dagger})^*(\, A^*BB^* - A^*AB^*\,)(B^{\dagger})^* = AA^{\dagger}B - AB^{\dagger}B.$$

Hence,

$$r(\, AA^{\dagger}B - AB^{\dagger}B\,) = r(\, A^*BB^* - A^*AB^*\,) = r(\, BA^*A - BB^*A\,).$$

Thus, (2.22) follows. $\square$

Theorem 2.7 *Let $A,\ B \in \mathbb{C}^{m\times n}$. Then*

$$\begin{aligned}
&\min_{A^{(1,2,3)},\, B^{(1,2,3)}} r(\, A^{(1,2,3)} - B^{(1,2,3)}\,)\\
&= \max\left\{ r[\,A,\ B\,] + r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix} - r\begin{bmatrix} A & B & 0 \\ A^*A & 0 & B^* \end{bmatrix},\right.\\
&\qquad\left. r[\,A,\ B\,] + r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix} - r\begin{bmatrix} A & 0 & B \\ 0 & A^* & B^*B \end{bmatrix}\right\}. \qquad (2.23)
\end{aligned}$$

Hence, A and B have a common $\{1,2,3\}$-inverse if and only if

$$r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix} = r\begin{bmatrix} A \\ B \end{bmatrix} \qquad (2.24)$$

and

$$r\begin{bmatrix} A & B & 0 \\ A^*A & 0 & B^* \end{bmatrix} = r\begin{bmatrix} A & 0 & B \\ 0 & A^* & B^*B \end{bmatrix} = r[\,A,\ B\,] + r\begin{bmatrix} A \\ B \end{bmatrix} \qquad (2.25)$$

hold, that is,

$$\mathscr{R}\begin{bmatrix} A^* \\ B^* \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A^*A \\ B^*B \end{bmatrix},\ \mathscr{R}\begin{bmatrix} A^* \\ B^* \\ 0 \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} A^*A \\ 0 \\ B \end{bmatrix} = \{0\} \text{ and } \mathscr{R}\begin{bmatrix} A^* \\ 0 \\ B^* \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} 0 \\ A \\ B^*B \end{bmatrix} = \{0\}.$$

Proof From (1.5), the general expression of $A^{(1,2,3)} - B^{(1,2,3)}$ is given by

$$A^{(1,2,3)} - B^{(1,2,3)} = A^{\dagger} - B^{\dagger} + F_AVAA^{\dagger} + F_BWBB^{\dagger}.$$

Hence, by (1.10)

$$\begin{aligned}
&\min_{A^{(1,2,3)},\, B^{(1,2,3)}} r(\, A^{(1,2,3)} - B^{(1,2,3)}\,)\\
&= \max_{V,\,W} r(\, A^{\dagger} - B^{\dagger} + F_AVAA^{\dagger} + F_BWBB^{\dagger}\,)\\
&= r\begin{bmatrix} A^{\dagger} - B^{\dagger} \\ AA^{\dagger} \\ BB^{\dagger} \end{bmatrix} + r[\,A^{\dagger} - B^{\dagger},\ F_A,\ F_B\,]\\
&\quad + \max\left\{ r\begin{bmatrix} A^{\dagger} - B^{\dagger} & F_A \\ BB^{\dagger} & 0 \end{bmatrix} - r\begin{bmatrix} A^{\dagger} - B^{\dagger} & F_A & F_B \\ BB^{\dagger} & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A^{\dagger} - B^{\dagger} & F_A \\ AA^{\dagger} & 0 \\ BB^{\dagger} & 0 \end{bmatrix},\right.\\
&\qquad\left. r\begin{bmatrix} A^{\dagger} - B^{\dagger} & F_B \\ AA^{\dagger} & 0 \end{bmatrix} - r\begin{bmatrix} A^{\dagger} - B^{\dagger} & F_A & F_B \\ AA^{\dagger} & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A^{\dagger} - B^{\dagger} & F_B \\ AA^{\dagger} & 0 \\ BB^{\dagger} & 0 \end{bmatrix}\right\}. \qquad (2.26)
\end{aligned}$$

Simplifying the ranks of the block matrices in (2.26) by (2.13) and elementary block matrix operations gives

$$r\begin{bmatrix} A^{\dagger} - B^{\dagger} \\ AA^{\dagger} \\ BB^{\dagger} \end{bmatrix} = r[\,A,\ B\,],$$

$$
\begin{aligned}
r[\, A^{\dagger} - B^{\dagger},\, F_A,\, F_B \,] &= r\begin{bmatrix} A^{\dagger} - B^{\dagger} & I_n & I_n \\ 0 & A & 0 \\ 0 & 0 & B \end{bmatrix} - r(A) - r(B) \\
&= r\begin{bmatrix} 0 & 0 & I_n \\ -AA^{\dagger} & A & 0 \\ BB^{\dagger} & -B & 0 \end{bmatrix} - r(A) - r(B) \\
&= n + r\begin{bmatrix} AA^{\dagger} & A \\ BB^{\dagger} & B \end{bmatrix} - r(A) - r(B) \\
&= n + r\begin{bmatrix} A^{*} & A^{*}A \\ B^{*} & B^{*}B \end{bmatrix} - r(A) - r(B) \quad (\text{by } (1.15)) \\
&= n + r\begin{bmatrix} A & B \\ A^{*}A & B^{*}B \end{bmatrix} - r(A) - r(B),
\end{aligned}
$$

$$
\begin{aligned}
r\begin{bmatrix} A^{\dagger} - B^{\dagger} & F_A \\ BB^{\dagger} & 0 \end{bmatrix} &= r\begin{bmatrix} A^{\dagger} & F_A \\ B^{*} & 0 \end{bmatrix} \\
&= r\begin{bmatrix} A^{\dagger} & I_n \\ B^{*} & 0 \\ 0 & A \end{bmatrix} - r(A) \\
&= r\begin{bmatrix} 0 & I_n \\ B^{*} & 0 \\ -AA^{\dagger} & 0 \end{bmatrix} - r(A) \\
&= n + r\begin{bmatrix} B^{*} \\ A^{*} \end{bmatrix} - r(A) = n + r[\, A,\, B \,] - r(A),
\end{aligned}
$$

$$
\begin{aligned}
r\begin{bmatrix} A^{\dagger} - B^{\dagger} & F_A & F_B \\ BB^{\dagger} & 0 & 0 \end{bmatrix} &= r\begin{bmatrix} A^{\dagger} & 0 & I_n \\ B^{*} & 0 & 0 \\ -AA^{\dagger} & A & 0 \\ 0 & -B & 0 \end{bmatrix} - r(A) - r(B) \\
&= n + r\begin{bmatrix} B^{\dagger} & 0 \\ AA^{\dagger} & A \\ 0 & B \end{bmatrix} - r(A) - r(B) \\
&= n + r\begin{bmatrix} B^{*} & 0 \\ A^{*} & A^{*}A \\ 0 & B \end{bmatrix} - r(A) - r(B) \\
&= n + r\begin{bmatrix} A & B & 0 \\ A^{*}A & 0 & B^{*} \end{bmatrix} - r(A) - r(B),
\end{aligned}
$$

$$
\begin{aligned}
r\begin{bmatrix} A^{\dagger} - B^{\dagger} & F_A \\ AA^{\dagger} & 0 \\ BB^{\dagger} & 0 \end{bmatrix} &= r\begin{bmatrix} 0 & F_A \\ AA^{\dagger} & 0 \\ BB^{\dagger} & 0 \end{bmatrix} \\
&= r\begin{bmatrix} AA^{\dagger} \\ BB^{\dagger} \end{bmatrix} + r(F_A) = r[\, A,\, B \,] + n - r(A).
\end{aligned}
$$

Substituting the above six equalities into (2.26) gives (2.23). □

The following two theorems can be shown similarly.

Theorem 2.8 *Let* $A,\ B \in \mathbb{C}^{m\times n}$. *Then*

$$\min_{A^{(1,2,4)},\, B^{(1,2,4)}} r(\, A^{(1,2,4)} - B^{(1,2,4)}\,) = \max\left\{ r\begin{bmatrix} A \\ B \end{bmatrix} + r\begin{bmatrix} A & AA^* \\ B & BB^* \end{bmatrix} - r\begin{bmatrix} A & AA^* \\ B & 0 \\ 0 & B^* \end{bmatrix},\right.$$
$$\left. r\begin{bmatrix} A \\ B \end{bmatrix} + r\begin{bmatrix} A & AA^* \\ B & BB^* \end{bmatrix} - r\begin{bmatrix} A & 0 \\ 0 & A^* \\ B & BB^* \end{bmatrix} \right\}.$$

Hence, A *and* B *have a common* $\{1,2,4\}$*-inverse if and only if*

$$r\begin{bmatrix} A & AA^* \\ B & BB^* \end{bmatrix} = r[\, A,\ B\,] \text{ and } r\begin{bmatrix} A & AA^* \\ B & 0 \\ 0 & B^* \end{bmatrix} = r\begin{bmatrix} A & 0 \\ 0 & A^* \\ B & BB^* \end{bmatrix} = r[\, A,\ B\,] + r\begin{bmatrix} A \\ B \end{bmatrix}$$

hold, that is,

$$\mathscr{R}\begin{bmatrix} A \\ B \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} AA^* \\ BB^* \end{bmatrix},\ \mathscr{R}\begin{bmatrix} A \\ B \\ 0 \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} AA^* \\ 0 \\ B^* \end{bmatrix} = \{0\} \text{ and } \mathscr{R}\begin{bmatrix} A \\ 0 \\ B \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} 0 \\ A^* \\ BB^* \end{bmatrix} = \{0\}.$$

Theorem 2.9 *Let* $A,\ B \in \mathbb{C}^{m\times n}$. *Then*

$$\min_{\substack{A^{(1,3,4)} \\ B^{(1,3,4)}}} r(\, A^{(1,3,4)} - B^{(1,3,4)}\,) = r\begin{bmatrix} A & AA^* \\ B & BB^* \end{bmatrix} + r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix}$$
$$+ \max\{ r(\, AA^*B - AB^*B\,) - r(\, A^*A - A^*B\,) - r(\, BB^* - AB^*\,),$$
$$r(\, BA^*A - BB^*A\,) - r(\, AA^* - BA^*\,) - r(\, B^*B - B^*A\,)\}.$$

3 The relations between $(A^*)^-$ and $(A^-)^*$

In this section, we investigate the relations between the two sets $\{(A^*)^-\}$ and $\{(A^-)^*\}$ consisting of g-inverses of A^* and A. A simple result on the Moore-Penrose inverse of A is

$$(A^*)^\dagger = (A^\dagger)^*.$$

In addition,

$$A^*(A^*)^\dagger = A^\dagger A, \quad (A^*)^\dagger A^* = AA^\dagger.$$

Applying these results to (1.1)–(1.7) gives

$$(A^-)^* = (A^*)^\dagger + V^*F_A + E_AW^*, \tag{3.1}$$
$$(A_r^-)^* = [\,(A^*)^\dagger + E_AW^*\,]A^*[\,(A^*)^\dagger + V^*F_A\,], \tag{3.2}$$
$$(A^{(1,3)})^* = (A^*)^\dagger + V^*F_A, \tag{3.3}$$
$$(A^{(1,4)})^* = (A^*)^\dagger + E_AW^*, \tag{3.4}$$
$$(A^{(1,2,3)})^* = (A^*)^\dagger + AA^\dagger V^*F_A, \tag{3.5}$$
$$(A^{(1,2,4)})^* = (A^*)^\dagger + E_AW^*A^\dagger A, \tag{3.6}$$
$$(A^{(1,3,4)})^* = (A^*)^\dagger + E_AV^*F_A, \tag{3.7}$$

and

$$(A^*)^- = (A^*)^\dagger + E_A V + W F_A, \tag{3.8}$$
$$(A^*)_r^- = [\,(A^*)^\dagger + E_A V\,)A^*[\,(A^*)^\dagger + W F_A\,], \tag{3.9}$$
$$(A^*)^{(1,3)} = (A^*)^\dagger + E_A V^*, \tag{3.10}$$
$$(A^*)^{(1,4)} = (A^*)^\dagger + W F_A, \tag{3.11}$$
$$(A^*)^{(1,2,3)} = (A^*)^\dagger + E_A V A^\dagger A, \tag{3.12}$$
$$(A^*)^{(1,2,4)} = (A^*)^\dagger + A A^\dagger W F_A, \tag{3.13}$$
$$(A^*)^{(1,3,4)} = (A^*)^\dagger + E_A V F_A. \tag{3.14}$$

Since the two matrices V and W are arbitrary, by comparing (3.1)–(3.7) and (3.8)–(3.14) we obtain the following results.

Theorem 3.1 *Let $A \in \mathbb{C}^{m\times m}$. Then*

(a) [6] $\{(A^*)^-\} = \{(A^-)^*\}$.

(b) $\{(A^*)_r^-\} = \{(A_r^-)^*\}$.

(c) $\{(A^*)^{(1,3)}\} = \{(A^{(1,4)})^*\}$.

(d) $\{(A^*)^{(1,4)}\} = \{(A^{(1,3)})^*\}$.

(e) $\{(A^*)^{(1,2,3)}\} = \{(A^{(1,2,4)})^*\}$.

(f) $\{(A^*)^{(1,2,4)}\} = \{(A^{(1,2,3)})^*\}$.

(g) $\{(A^*)^{(1,3,4)}\} = \{(A^{(1,3,4)})^*\}$.

If A^* is Hermitian, i.e., $A^* = A$, we obtain the following from Theorem 3.1.

Corollary 3.2 *Let $A \in \mathbb{C}^{m\times m}$ be Hermitian. Then*

(a) $\{A^-\} = \{(A^-)^*\}$.

(b) $\{A_r^-\} = \{(A_r^-)^*\}$.

(c) $\{A^{(1,3)}\} = \{(A^{(1,4)})^*\}$.

(d) $\{A^{(1,4)}\} = \{(A^{(1,3)})^*\}$.

(e) $\{A^{(1,2,3)}\} = \{(A^{(1,2,4)})^*\}$.

(f) $\{A^{(1,2,4)}\} = \{(A^{(1,2,3)})^*\}$.

(g) $\{A^{(1,3,4)}\} = \{(A^{(1,3,4)})^*\}$.

Hermitian g-inverses of a general square matrix do not necessarily exist. If, however, A is Hermitian, A_h^- exists and their general expressions are given as follows.

Theorem 3.3 *Let $A \in \mathbb{C}^{m\times m}$ be Hermitian. Then the general expressions of the Hermitian g-inverses A_h^-, $A_h^{(1,2)}$, $A_h^{(1,3)}$, $A_h^{(1,4)}$, $A_h^{(1,2,3)}$, $A_h^{(1,2,4)}$ and $A_h^{(1,3,4)}$ are given by*

(a) $A_h^- = A^\dagger + E_A V + V^* E_A$, *where V is arbitrary.*

(b) $A_h^{(1,2)} = (A^\dagger + E_A V) A (A^\dagger + V^* E_A)$, *where* V *is arbitrary.*

(c) $A_h^{(1,3)} = A^\dagger + E_A U E_A$, *where* $U = U^*$ *is arbitrary.*

(d) $A_h^{(1,4)} = A^\dagger + E_A U E_A$, *where* $U = U^*$ *is arbitrary.*

(e) $A_h^{(1,3,4)} = A^\dagger + E_A U E_A$, *where* $U = U^*$ *is arbitrary.*

(f) $A_h^{(1,2,3)} = A_h^{(1,2,4)} = A^\dagger$.

Proof Since A is Hermitian, $A^\dagger$ is Hermitian, too. Hence A_h^- in (a) is Hermitian. Also suppose X_0 is any Hermitian g-inverse of A, i.e., $AX_0A = A$ and $X_0^* = X_0$. In this case, let $V = (X_0 + X_0AA^\dagger)/2$ in (a). Then

$$A_h^- = A^\dagger + \frac{1}{2}E_A(X_0 + X_0AA^\dagger) + \frac{1}{2}(X_0 + AA^\dagger X_0)E_A = A^\dagger + X_0 - A^\dagger = X_0.$$

Hence (a) is the general expression of A_h^-. The matrix $A_h^{(1,2)}$ in (b) is a Hermitian $\{1,2\}$-inverse of A. Also suppose X_0 is any Hermitian $\{1,2\}$-inverse of A, i.e., $AX_0A = A$, $X_0AX_0 = X_0$ and $X_0^* = X_0$. In this case, let $V = X_0$ in $A_h^{(1,2)}$. Then

$$A_h^{(1,2)} = (A^\dagger + E_A X_0) A (A^\dagger + X_0 E_A) = X_0 A (A^\dagger + X_0 E_A) = X_0 A X_0 = X_0.$$

Hence (b) is the general expression of $A_h^{(1,2)}$. The proofs of (c)–(f) are left to the reader. □

From the general expressions of the Hermitian g-inverses in Theorem 3.3, one can find various properties of these g-inverses, for example, the extremal ranks of the g-inverses; common Hermitian g-inverses of two Hermitian matrices.

If A and B are Hermitian or normal, or they satisfy $B = A^k$, $B = I - A$, or $A^*B = 0$, the results in Theorems 2.1–2.9 can be simplified further.

Many consequences can be derived from Theorems 2.1–2.9. For instance, replacing B with $A + B$, where B is a perturbation matrix, one can obtain a set of results on common g-inverses of A and $A + B$. For example,

(a) $A + B$ and A have a common $\{1\}$-inverse if and only if

$$r(A + B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] - r(A) - r(B).$$

(b) The set inclusion $\{(A + B)^-\} \subseteq \{A^-\}$ holds if and only if $r(A + B) = r(A) + r(B)$.

(c) $\{(A + B)^-\} = \{A^-\}$ if and only if $B = 0$.

(d) $\{(A + B)^-\} \cap \{A^-\} = \{0\}$ if and only if $r(A + B) > r\begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] - r(A) - r(B)$.

Moreover, a variety of results on common g-inverses of partitioned matrices can be derived. For instance, let $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ and $N = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$. Then

$$\min_{M^-, N^-} r(M^- - N^-) = r(M) - r\begin{bmatrix} B \\ C \end{bmatrix} - r[A, B] + r(B).$$

$$\max_{M^-} r(N - NM^-N) = r(A) + r(B) + r(C) - r(M).$$

Hence, M and N have a common $\{1\}$-inverse if and only if

$$r(M) = r\begin{bmatrix} B \\ C \end{bmatrix} + r[\,A,\, B\,] - r(B);$$

$\{M^-\} \subseteq \{N^-\}$ if and only if $r(M) = r(A) + r(B) + r(C)$, i.e., $\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$ and $\mathscr{R}(B^*) \cap \mathscr{R}(C^*) = \{0\}$.

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