Nonnegative determinant of a rectangular matrix: Its definition and applications to multivariate analysis[1]

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ABSTRACT

It is well known that the determinant of a matrix can only be defined for a square matrix. In this paper, we propose a new definition of the determinant of a rectangular matrix and examine its properties. We apply these properties to squared canonical correlation coefficients, and to squared partial canonical correlation coefficients. The proposed definition of the determinant of a rectangular matrix allows an easy and straightforward decomposition of the likelihood ratio when given sets of variables are partitioned into row block matrices. The last section describes a general theorem on redundancies among variables measured in terms of the likelihood ratio of a partitioned matrix.

1 Introduction

It is well known that the determinant of a matrix in the usual sense can only be defined for a square matrix. Farebrother (1997) noted, however, that Cullis (1913-1925) published three large volumes on matrix algebra, in which he introduced the determinant of a rectangular matrix, which he called "determinoid." Cullis' notion of determinoid, however, does not necessarily reflect the geometrical structure of elements embedded in a rectangular matrix, and consequently its use in statistics is quite limited.

In this paper, we propose a new definition of the determinant of a rectangular matrix from a geometrical point of view and examine its properties. We then apply the notion to standard multivariate analysis methods. Let $\mathbf{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p]$ denote an $n \times p$ $(p \neq n)$ rectangular matrix. The determinant of $\mathbf{X}$ cannot be defined in the usual sense. Instead, we define $\det(\mathbf{X}'\mathbf{X})$, which is equal to n times the determinant of the covariance matrix of $\mathbf{X}$ if $\mathbf{X}$ is a columnwise centered data matrix (i.e., a matrix of deviation scores from column means). Here, $\det(\mathbf{X}'\mathbf{X})/n$ is called the generalized variance of $\mathbf{X}$, which is equal to the squared volume of the p dimensional simplex defined by p column vectors of $\mathbf{X}$ (Takeuchi, Yanai & Mukherjee, 1982, pp. 76-78).

We first apply these properties to squared canonical correlation coefficients, and then to squared partial canonical correlation coefficients. The cases of squared correlation coefficients, squared partial correlation coefficients, and squared multiple correlation coefficients follow as special cases. It will further be shown that the proposed definition of the determinant of a rectangular matrix renders a straightforward decomposition of the likelihood ratio, when the given sets of variables, $\mathbf{X}$ and $\mathbf{Y}$, are partitioned into row block matrices, $\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_p]$ and $\mathbf{Y} = [\mathbf{Y}_1, \ldots, \mathbf{Y}_q]$. The final section describes a general theorem on redundancies among variables measured in terms of the likelihood ratio of a partitioned matrix.

In the remaining part of this section, we give some notations and some preliminary results concerning orthogonal projectors useful in this paper. For a given $n \times m$ matrix $\mathbf{A}$, we let $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$ denote the range space and the null space of $\mathbf{A}$. We use $\mathbf{P}_A$ to denote the orthogonal projector onto $\mathcal{R}(\mathbf{A})$, and $\mathbf{Q}_A = \mathbf{I}_n - \mathbf{P}_A$ to denote the orthogonal projector onto $\mathcal{N}(\mathbf{A}') = \mathcal{R}(\mathbf{A})^{\perp}$, the null space of $\mathbf{A}'$ and the orthogonal complement subspace of $\mathcal{R}(\mathbf{A})$. It is well known that

$$\mathbf{P}_A = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}' \text{ and } \mathbf{Q}_A = \mathbf{I}_n - \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'.$$

For a definition of projectors $\mathbf{A}'\mathbf{A}$ could be singular, and the regular inverse of $\mathbf{A}'\mathbf{A}$ may be replaced by a generalized inverse (g-inverse). In the above definition of projectors , however, $\mathbf{A}'\mathbf{A}$ is assumed to be nonsingular, as it is usually assumed in this paper.

We first give a lemma stating important decompositions of orthogonal projectors onto $\mathcal{R}([\mathbf{X},\mathbf{Y}])$ and its ortho-complement subspace.

Lemma 1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be matrices of the same row order. Further, let* $\mathbf{P}_{[X,Y]}$ *be the orthogonal projector onto* $\mathcal{R}([\mathbf{X},\mathbf{Y}])$*, and let* $\mathbf{Q}_{[X,Y]} = \mathbf{I}_n - \mathbf{P}_{[X,Y]}$*. Then,*

$$\mathbf{P}_{[X,Y]} = \mathbf{P}_X + \mathbf{P}_{Q_X Y} = \mathbf{P}_Y + \mathbf{P}_{Q_Y X}, \tag{1}$$

$$\mathbf{Q}_{[X,Y]} = \mathbf{Q}_{Q_X Y}\mathbf{Q}_X = \mathbf{Q}_{Q_Y X}\mathbf{Q}_Y. \tag{2}$$

The first decomposition, (1), was given by Rao & Yanai (1979; Theorem 6), and the second, (2) by Yanai & Puntanen (1993; Lemma 1).

The following lemma gives a well-known formula on the decomposition of the determinant of a partitioned square matrix.

Lemma 2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times p$ *and* $n \times q$ *columnwise nonsingular matrices. Let*

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}'\mathbf{A} & \mathbf{A}'\mathbf{B} \\ \mathbf{B}'\mathbf{A} & \mathbf{B}'\mathbf{B} \end{bmatrix}. \tag{3}$$

Then,

$$\det(\mathbf{C}) = \det(\mathbf{A}'\mathbf{A})\det(\mathbf{B}'\mathbf{B} - \mathbf{B}'\mathbf{P}_A\mathbf{B}) = \det(\mathbf{A}'\mathbf{A})\det(\mathbf{B}'\mathbf{Q}_A\mathbf{B}) \tag{4}$$

$$= \det(\mathbf{B}'\mathbf{B})\det(\mathbf{A}'\mathbf{A} - \mathbf{A}'\mathbf{P}_B\mathbf{A}) = \det(\mathbf{B}'\mathbf{B})\det(\mathbf{A}'\mathbf{Q}_B\mathbf{A}) \tag{5}$$

where $\mathbf{P}_A = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'$ *and* $\mathbf{P}_B = \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'$ *are orthogonal projectors onto* $\mathcal{R}(\mathbf{A})$ *and* $\mathcal{R}(\mathbf{B})$*, respectively, while* $\mathbf{Q}_A = \mathbf{I}_n - \mathbf{P}_A$ *and* $\mathbf{Q}_B = \mathbf{I}_n - \mathbf{P}_B$ *are their orthogonal complements, that is, the projectors onto* $\mathcal{R}(\mathbf{A})^\perp$ *and* $\mathcal{R}(\mathbf{B})^\perp$*, respectively.*

Observe that $\mathbf{B}'\mathbf{Q}_A\mathbf{B}$ is called the Schur complement of $\mathbf{A}'\mathbf{A}$ in $\mathbf{C}$, while $\mathbf{A}'\mathbf{Q}_B\mathbf{A}$ is called the Schur complement of $\mathbf{B}'\mathbf{B}$ in $\mathbf{C}$.
The readers interested in further discussions on orthogonal projectors, see Rao & Yanai (1979), Yanai (1981), and Takeuchi, Yanai, & Mukherjee (1982, pp. 27-44).

2 Definition of the determinant of a rectangular matrix and its properties

Definition 1. *For a given matrix* $\mathbf{A}$ *of order* $n \times m$*, we define the determinant of a rectangular matrix as the square root of the determinant of* $\mathbf{A}'\mathbf{A}$*. That is,*

$$\mathrm{ndet}(\mathbf{A}) = \sqrt{\det(\mathbf{A}'\mathbf{A})}\ . \tag{6}$$

The following properties hold regarding the determinant of a rectangular matrix just defined. Let $\mathbf{A}$ be an $n \times m$ $(n \neq m)$ matrix.

Property 1. $\mathrm{ndet}(\mathbf{A}) \geq 0$ and the equality holds when $\mathcal{N}(\mathbf{A})$ is non-null.
Property 2. $\mathrm{ndet}(-\mathbf{A}) \;=\; \mathrm{ndet}(\mathbf{A})$.
Property 3. If $n = m$, then $\mathrm{ndet}(\mathbf{A}) \;=\; \mathrm{abs}(\det(\mathbf{A}))$.
Property 4. $\mathrm{ndet}(k\mathbf{A}) \;=\; (\mathrm{abs}(k))^m\,\mathrm{ndet}(\mathbf{A})$.
Property 5. Let the kth singular value of $\mathbf{A}$ be denoted by $\mu_k(\mathbf{A})$. Then,

$$\mathrm{ndet}(\mathbf{A}) \;=\; \prod_{k=1}^{m} \mu_k(\mathbf{A}) \;. \tag{7}$$

Note 1. Note that according to (6), $\mathrm{ndet}(\mathbf{A}) \neq \mathrm{ndet}(\mathbf{A}')$. One idea to ensure $\mathrm{ndet}(\mathbf{A}) = \mathrm{ndet}(\mathbf{A}')$ is to modify m in (7) to $\mathrm{Min}(n, m)$ and use it as the definition of $\mathrm{ndet}(\mathbf{A})$. However, the results that follow from (6) are considerably richer in statistics, where $\mathbf{A}$ is n by m, and often $\mathrm{rank}(\mathbf{A}) = m$.

Property 6. If matrices $\mathbf{A}$ and $\mathbf{B}$ are of orders $n \times p$ and $n \times q$, $\mathrm{ndet}([\mathbf{A}, \mathbf{B}]) = \mathrm{ndet}([\mathbf{B}, \mathbf{A}])$.
Property 7. If matrices $\mathbf{A}$ and $\mathbf{C}$ are of orders $m \times n$ and $n \times n$, then

$$\mathrm{ndet}(\mathbf{AC}) = \mathrm{ndet}(\mathbf{A})\,\mathrm{ndet}(\mathbf{C}') \;. \tag{8}$$

This follows from $\mathrm{ndet}(\mathbf{AC}) = \sqrt{\det(\mathbf{C}'\mathbf{A}'\mathbf{AC})} = \sqrt{\det(\mathbf{A}'\mathbf{A})}\sqrt{\det(\mathbf{CC}')}$.

Property 8. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times m$ matrices, and let $\mathbf{D}_1$ and $\mathbf{D}_2$ be square matrices of orders m. Let $[\mathbf{A}, \mathbf{B}]$ indicate a row block matrix. Then, the following relations hold.

$$\mathrm{ndet}([\mathbf{A}, \mathbf{B} + \mathbf{AD}_1]) \;=\; \mathrm{ndet}([\mathbf{A}, \mathbf{B}]) \tag{9}$$

$$\mathrm{ndet}([\mathbf{A} + \mathbf{BD}_2, \mathbf{B}]) \;=\; \mathrm{ndet}([\mathbf{A}, \mathbf{B}]) \;. \tag{10}$$

This immediately follows from Lemma 1 and Definition 1. As special cases of Property 8, we note the following.

Note 2. $\mathrm{ndet}([\mathbf{A} + \mathbf{B}, \mathbf{B}]) \;=\; \mathrm{ndet}([\mathbf{A}, \mathbf{A} + \mathbf{B}]) \;=\; \mathrm{ndet}([\mathbf{A}, \mathbf{B}])$.

Using Property 8, we can establish the following lemma.

Lemma 3. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices of orders* $n \times p$ *and* $n \times q$*, respectively. Then,*

$$\mathrm{ndet}([\mathbf{A}, \mathbf{B}]) \;=\; \mathrm{ndet}(\mathbf{A})\,\mathrm{ndet}(\mathbf{Q}_A\mathbf{B}) \;=\; \mathrm{ndet}(\mathbf{B})\,\mathrm{ndet}(\mathbf{Q}_B\mathbf{A}), \tag{11}$$

$$\mathrm{ndet}([\mathbf{A}, \mathbf{B}]) \;\leq\; \mathrm{ndet}(\mathbf{A})\,\mathrm{ndet}(\mathbf{B}), \tag{12}$$

$$\mathrm{ndet}([\mathbf{A}, \mathbf{B}]) \;=\; \mathrm{ndet}(\mathbf{A})\,\mathrm{ndet}(\mathbf{B}) \;\Leftrightarrow\; \mathbf{A}'\mathbf{B} \;=\; \mathbf{O}. \tag{13}$$

Proof of (11). Formula (11) easily follows from (9) by substituting $-(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{B}$ for $\mathbf{D}_1$, or from (10) by substituting $-(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{A}$ for $\mathbf{D}_1$, since $\mathbf{B} + \mathbf{AD}_1 = \mathbf{B} - \mathbf{A}(\mathbf{A}'\mathbf{A})^{-}\mathbf{A}'\mathbf{B} = \mathbf{Q}_A\mathbf{B}$, and $\mathbf{A} + \mathbf{BD}_2 = \mathbf{A} - \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{A} = \mathbf{Q}_B\mathbf{A}$.

To prove (12), we need the following property.

Property 9. Suppose that there exist two distinct matrices, $\mathbf{X}$ and $\mathbf{Y}$, both of order $n \times p$, such that $\mathbf{A} = \mathbf{X}'\mathbf{X}$ and $\mathbf{B} = \mathbf{Y}'\mathbf{Y}$. If $\mathbf{A} - \mathbf{B} \geq \mathbf{O}$, then $\mathrm{ndet}(\mathbf{X}) \geq \mathrm{ndet}(\mathbf{Y})$.

Proof. Property 9 holds, since if $\mathbf{A} \geq \mathbf{B} \geq \mathbf{O}$, $\lambda_k(\mathbf{A}) \geq \lambda_k(\mathbf{B})$, which implies $\mu_k(\mathbf{A}) \geq \mu_k(\mathbf{B})$ for

$k = 1, \ldots, p$. Using Property 5, we can prove the desired result.

Proof of (12) and (13) of Lemma 3. To prove (12), observe that $\mathbf{A}'\mathbf{A} = \mathbf{A}'\mathbf{P}_B\mathbf{A} + \mathbf{A}'\mathbf{Q}_B\mathbf{A} \geq \mathbf{A}'\mathbf{Q}_B\mathbf{A} \geq \mathbf{O}$, leading to $\mathrm{ndet}(\mathbf{A}) \geq \mathrm{ndet}(\mathbf{Q}_B\mathbf{A})$, which establishes a proof by (11) of Lemma 3. To prove (13), we note that $\mathbf{A}'\mathbf{B} = \mathbf{O}$ implies $\mathrm{ndet}([\mathbf{A}, \mathbf{B}]) = \mathrm{ndet}(\mathbf{A})\,\mathrm{ndet}(\mathbf{B})$. To prove the reverse, we first obtain

$$\mathrm{ndet}(\mathbf{B}) = \mathrm{ndet}(\mathbf{Q}_A\mathbf{B}) \Rightarrow \det(\mathbf{B}'\mathbf{Q}_A\mathbf{B})/\det(\mathbf{B}'\mathbf{B}) = 1,$$

where $\mathbf{C} = (\mathbf{B}'\mathbf{B})^{-1/2}(\mathbf{B}'\mathbf{A})(\mathbf{A}'\mathbf{A})^{-1/2}$, since $\det(\mathbf{I}_p + \mathbf{A}\mathbf{B}') = \det(\mathbf{I}_q + \mathbf{B}'\mathbf{A})$. It follows that $0 \leq \lambda_k(\mathbf{C}'\mathbf{C}) \leq 1$ for $k = 1, \ldots, \mathrm{rank}(\mathbf{C})$. Then, $\det(\mathbf{I}_p - \mathbf{C}'\mathbf{C}) = 1$ implies $\mathbf{C} = \mathbf{O}$, establishing $\mathbf{B}'\mathbf{A} = \mathbf{O}$. (Q.E.D.)

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are matrices of orders $n \times p$ and $n \times q$, respectively, and of full column rank.

Definition 2. *We introduce*

$$Re(\mathbf{A}, \mathbf{B}) \;=\; \mathrm{ndet}([\mathbf{A}, \mathbf{B}])/\big(\mathrm{ndet}(\mathbf{A})\,\mathrm{ndet}(\mathbf{B})\big). \tag{14}$$

From (12) of Lemma 3, we have $0 \leq Re([\mathbf{A}, \mathbf{B}]) \leq 1$, and from (13) of Lemma 3, a necessary and sufficient condition for $Re([\mathbf{A}, \mathbf{B}]) = 1$ is $\mathbf{A}'\mathbf{B} = \mathbf{O}$. Observe that (14) can be expanded as

$$Re(\mathbf{A}, \mathbf{B}) \;=\; \sqrt{\det\left(\begin{bmatrix} \mathbf{A}'\mathbf{A} & \mathbf{A}'\mathbf{B} \\ \mathbf{B}'\mathbf{A} & \mathbf{B}'\mathbf{B} \end{bmatrix}\right)} \Big/ \big(\sqrt{\det(\mathbf{A}'\mathbf{A})}\sqrt{\det(\mathbf{B}'\mathbf{B})}\big), \tag{15}$$

which is the square root of the likelihood ratio of the variance covariance matrix of $[\mathbf{A}, \mathbf{B}]$, if both $\mathbf{A}$ and $\mathbf{B}$ are columnwise centered.

Property 10. Suppose $\mathbf{T}$ and $\mathbf{S}$ are square matrices of orders p and q, respectively, such that $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{AT})$ and $\mathrm{rank}(\mathbf{B}) = \mathrm{rank}(\mathbf{BS})$. Then,

$$Re(\mathbf{AT}, \mathbf{BS}) \;=\; Re(\mathbf{A}, \mathbf{B}).$$

When m matrices, $\mathbf{A}_1, \ldots, \mathbf{A}_m$, have the same number of rows, we can extend Lemma 3 to the following three corollaries.

Corollary 1.

$$\mathrm{ndet}([\mathbf{A}_1, \ldots, \mathbf{A}_m]) \;=\; \mathrm{ndet}(\mathbf{A}_1) \prod_{j=2}^{m} \mathrm{ndet}(\mathbf{Q}_{[A_1,\ldots,A_{j-1}]}\mathbf{A}_j)\ . \tag{16}$$

Corollary 2.

$$\mathrm{ndet}(\mathbf{A}_1, \ldots, \mathbf{A}_m) \;\leq\; \prod_{j=1}^{m} \mathrm{ndet}(\mathbf{A}_j)\ . \tag{17}$$

Corollary 3.

$$\mathbf{A}_i'\mathbf{A}_j = \mathbf{O}\ (i \neq j) \;\Leftrightarrow\; \mathrm{ndet}([\mathbf{A}_1, \ldots, \mathbf{A}_m]) = \prod_{j=1}^{m} \mathrm{ndet}(\mathbf{A}_j)\ . \tag{18}$$

Note 3. If each $\mathbf{A}_j$ consists of a single standardized vector (with zero mean and unit variance), then $\big(\mathrm{ndet}([\mathbf{A}_1,\ldots,\mathbf{A}_m])\big)^2/n = \det(\mathbf{R}) \leq 1$ where $\mathbf{R}$ is the correlation matrix among m variables.

Note 4. If in Corollary 2, each $\mathbf{A}_j$ consists of a single vector $\boldsymbol{a}_j$, i.e., $\mathbf{A} = [\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m]$, then

$$\mathrm{ndet}(\mathbf{A}) \;\leq\; \prod_{j=1}^{m}\sqrt{\sum_{i=1}^{n} a_{ij}^2}\ . \tag{19}$$

The above inequality is an extension of the Hadamard inequality in the sense that (19) holds even if $\mathbf{A}$ is a rectangular matrix. If $\mathbf{A}$ is a square matrix, it reduces to the original Hadamard inequality.

Proofs of Corollaries 1 and 2. We first give a proof of Corollary 1. In order to represent $\mathrm{ndet}([\mathbf{A}_1,\ldots,\mathbf{A}_m])$ as the product of two nonnegative determinants, we may use the following decomposition,

$$\begin{aligned}\mathrm{ndet}([\mathbf{A}_1,\ldots,\mathbf{A}_m]) &= \mathrm{ndet}\big([[\mathbf{A}_1,\ldots,\mathbf{A}_{j-1}],\mathbf{A}_j]\big)\\ &= \mathrm{ndet}([\mathbf{A}_1,\ldots,\mathbf{A}_{j-1}])\,\mathrm{ndet}\big(\mathbf{Q}_{[A_1,\ldots,A_{j-1}]}\mathbf{A}_j\big)\ .\end{aligned}$$

Repeating this formula from $j=2$ up to m, we have $m-1$ equations. Multiplying these equations, we obtain (16). To prove Corollary 2, we note

$$\mathrm{ndet}\big(\mathbf{Q}_{[A_1,\ldots,A_{j-1}]}\mathbf{A}_j\big) \leq \mathrm{ndet}(\mathbf{A}_j).$$

Furthermore, when $m>2$, (14) can be extended to

$$Re_{(m)}(\mathbf{A}_1,\ldots,\mathbf{A}_m) \;=\; \mathrm{ndet}([\mathbf{A}_1,\ldots,\mathbf{A}_m])\Big/\prod_{j=1}^{m}\mathrm{ndet}(\mathbf{A}_j)\ . \tag{20}$$

For simplicity, we write $Re(\mathbf{A}_1,\mathbf{A}_2)$ instead of $Re_{(2)}(\mathbf{A}_1,\mathbf{A}_2)$ hereafter for $m=2$. In view of the definition (20), Corollary 2 can be written as $0 \leq Re_{(m)}(\mathbf{A}_1,\ldots,\mathbf{A}_m) \leq 1$, while Corollary 3 can be written as

$$\mathbf{A}_i'\mathbf{A}_j = \mathbf{O}\ \ (i\neq j)\ \Leftrightarrow\ Re_{(m)}(\mathbf{A}_1,\ldots,\mathbf{A}_m) = 1\ .$$

(Q.E.D.)

Let $\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix}$ denote a column block matrix. Then,

Property 11.

$$\mathrm{ndet}\left(\begin{bmatrix}\mathbf{A}\\ \mathbf{A}\end{bmatrix}\right) = \mathrm{ndet}\left(\begin{bmatrix}\mathbf{A}\\ -\mathbf{A}\end{bmatrix}\right) = \sqrt{2}\,\mathrm{ndet}(\mathbf{A})\ .$$

We now state Theorem 1.

Theorem 1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ $n\times m$ *matrices. Further, let* $\mathbf{C} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B} & \mathbf{A}\end{bmatrix}$ *. Then,*

$$\mathrm{ndet}(\mathbf{C}) \;=\; \mathrm{ndet}(\mathbf{A}+\mathbf{B})\,\mathrm{ndet}(\mathbf{A}-\mathbf{B})\ . \tag{21}$$

Proof. From (11) of Lemma 3, we have

$$\mathrm{ndet}(\mathbf{C}) \;=\; \mathrm{ndet}\left(\begin{bmatrix} \mathbf{A}+\mathbf{B} & \mathbf{B} \\ \mathbf{A}+\mathbf{B} & \mathbf{A} \end{bmatrix}\right) \;=\; \mathrm{ndet}(\mathbf{X})\,\mathrm{ndet}\left(\mathbf{Q}_X \begin{bmatrix} \mathbf{B} \\ \mathbf{A} \end{bmatrix}\right) ,$$

where $\mathbf{X} = \begin{bmatrix} \mathbf{A}+\mathbf{B} \\ \mathbf{A}+\mathbf{B} \end{bmatrix}$. Observe that

$$\mathbf{Q}_X \begin{bmatrix} \mathbf{B} \\ \mathbf{A} \end{bmatrix} \;=\; (\mathbf{I}_{2n} - \mathbf{P}_X) \begin{bmatrix} \mathbf{B} \\ \mathbf{A} \end{bmatrix} \;=\; \begin{bmatrix} \mathbf{B} \\ \mathbf{A} \end{bmatrix} - \frac{1}{2}\mathbf{X} \;=\; \frac{1}{2} \begin{bmatrix} -(\mathbf{A}-\mathbf{B}) \\ (\mathbf{A}-\mathbf{B}) \end{bmatrix} ,$$

establishing (21) by Property 2 and Property 11.

If $\mathbf{A}$ and $\mathbf{B}$ in $\mathbf{C}$ are square matrices of the same order, it is well known that

$$\det(\mathbf{C}) \;=\; \det(\mathbf{A}+\mathbf{B})\det(\mathbf{A}-\mathbf{B}) \;.$$

It is interesting to note that the same type of decomposition as in (21) holds even if $\mathbf{C}$ is a rectangular matrix. Based on Theorem 1, we make the following remark.

Note 5. Let $\mathbf{A} = (\boldsymbol{x})$ and $\mathbf{B} = (\boldsymbol{y})$ be columnwise centered vectors. Further, let s_x^2, s_y^2 and s_{xy} denote variances of x and y, and covariance of x and y, respectively. Then,

$$\frac{\mathrm{ndet}\left(\begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \\ \boldsymbol{y} & \boldsymbol{x} \end{bmatrix}\right)}{\mathrm{ndet}\left(\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}\right)\ \mathrm{ndet}\left(\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{x} \end{bmatrix}\right)} \;=\; 1 - \frac{2s_{xy}}{s_x^2 + s_y^2} \;. \tag{22}$$

Observe that $\dfrac{2s_{xy}}{s_x^2 + s_y^2}$ is the squared intraclass correlation between $\boldsymbol{x}$ and $\boldsymbol{y}$.

3 Applications to multivariate analysis

In this section, we give some applications of the results presented in the previous section and illustrate the usefulness of the nonnegative determinant of a rectangular matrix for representing various methods of multivariate analysis. We first give a theorem on various equivalent representations of the nonnegative determinant of the likelihood ratio of two sets of variables.

Theorem 2. *Let $\mathbf{X}$ and $\mathbf{Y}$ be columnwise centered matrices of orders $n \times p$ and $n \times q$, respectively, and of full column rank. Further, let $cc_k(\mathbf{X}, \mathbf{Y})$ denote the kth largest canonical correlation between $\mathbf{X}$ and $\mathbf{Y}$. Then, the following quantities are all equal.*

(a) $\dfrac{\mathrm{ndet}([\mathbf{X}, \mathbf{Y}])}{\mathrm{ndet}(\mathbf{X})\,\mathrm{ndet}(\mathbf{Y})}$, (b) $\dfrac{\mathrm{ndet}(\mathbf{Q}_X\mathbf{Y})}{\mathrm{ndet}(\mathbf{Y})}$, (c) $\dfrac{\mathrm{ndet}(\mathbf{Q}_Y\mathbf{X})}{\mathrm{ndet}(\mathbf{X})}$,

(d) $\sqrt{\dfrac{\det(\mathbf{Y}'\mathbf{Q}_X\mathbf{Y})}{\det(\mathbf{Y}'\mathbf{Y})}} \;=\; \sqrt{\det(\mathbf{I}_q - \mathbf{Y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}(\mathbf{Y}'\mathbf{Y})^{-1})}$,

(e) $\sqrt{\det(\mathbf{I}_n - \mathbf{P}_X\mathbf{P}_Y)}$, (f) $\sqrt{\displaystyle\prod_{k=1}^{Min(p,q)} \left(1 - cc_k^2(\mathbf{X}, \mathbf{Y})\right)}$,

where $cc_k(\mathbf{X},\mathbf{Y})$ is the kth canonical correlation coefficient between $\mathbf{X}$ and $\mathbf{Y}$.

Proof. Proofs of (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c) can be easily established, using the formulae given in (11). A proof of the equivalence between (b) and (d) is also straightforward. (Q.E.D.)

Observe that any of the six quantities given in Theorem 2 is equivalent to what Hotelling (1936) called the vector coefficient of alienation.

The following two corollaries are special cases of Theorem 1, in which both $\mathbf{X}$ and $\mathbf{Y}$ consist of a single vector (Corollary 4), and in which $\mathbf{X}$ is a matrix while $\mathbf{Y}$ is a single vector (Corollary 5).

Corollary 4. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two columnwise centered n-component vectors. Further, let $r(\boldsymbol{x},\boldsymbol{y})$ denote the correlation coefficient between $\boldsymbol{x}$ and $\boldsymbol{y}$, and let $\theta(\boldsymbol{x},\boldsymbol{y})$ denote the angle between vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Then, the following quantities are equal.*

(a) $\dfrac{\mathrm{ndet}([\mathbf{x},\mathbf{y}])}{\mathrm{ndet}(\boldsymbol{x})\,\mathrm{ndet}(\boldsymbol{y})}$, (b) $\dfrac{\mathrm{ndet}(\mathbf{Q}_x\boldsymbol{y})}{\mathrm{ndet}(\boldsymbol{y})}$, (c) $\dfrac{\mathrm{ndet}(\mathbf{Q}_y\boldsymbol{x})}{\mathrm{ndet}(\boldsymbol{x})}$,

(d) $\sqrt{\dfrac{\boldsymbol{y}'\mathbf{Q}_x\boldsymbol{y}}{\boldsymbol{y}'\boldsymbol{y}}}$, (e) $\sqrt{\dfrac{\boldsymbol{x}'\mathbf{Q}_y\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{x}}}$, (f) $\sqrt{1-r^2(\boldsymbol{x},\boldsymbol{y})}$,

(g) $|\sin\theta(\boldsymbol{x},\boldsymbol{y})|$.

To see the equivalence between (f) and (g), we note $r(\boldsymbol{x},\boldsymbol{y}) = \cos\theta(\boldsymbol{x},\boldsymbol{y})$.

Corollary 5. *Let $\mathbf{X} = [\boldsymbol{x}_1,\ldots,\boldsymbol{x}_p]$ be an $n\times p$ columnwise centered matrix of predictor variables, and let $\boldsymbol{y}$ be a columnwise centered n-dimensional vector of the criterion variable. Let $mc(\mathbf{X},\boldsymbol{y})$ denote the multiple correlation coefficient obtained by regressing $\boldsymbol{y}$ onto $\mathbf{X}$. Then, the following four quantities are equal.*

(a) $\dfrac{\mathrm{ndet}([\mathbf{X},\boldsymbol{y}])}{\mathrm{ndet}(\mathbf{X})\,\mathrm{ndet}(\boldsymbol{y})}$, (b) $\dfrac{\mathrm{ndet}(\mathbf{Q}_X\boldsymbol{y})}{\mathrm{ndet}(\boldsymbol{y})}$,

(c) $\sqrt{\dfrac{\boldsymbol{y}'\mathbf{Q}_X\boldsymbol{y}}{\boldsymbol{y}'\boldsymbol{y}}}$, (d) $\sqrt{1-mc^2(\mathbf{X},\boldsymbol{y})}$.

Theorem 3. *In addition to matrices $\mathbf{X}$ and $\mathbf{Y}$ given in Theorem 2, add a third matrix $\mathbf{Z}$ of order $n\times r$, again assumed columnwise centered. Further, assume that both $\mathbf{Q}_Z\mathbf{X}$ and $\mathbf{Q}_Z\mathbf{Y}$ are of full column rank. Then, the following seven quantities are equal.*

(a) $\dfrac{\mathrm{ndet}([\mathbf{X},\mathbf{Y},\mathbf{Z}])\,\mathrm{ndet}(\mathbf{Z})}{\mathrm{ndet}([\mathbf{X},\mathbf{Z}])\,\mathrm{ndet}([\mathbf{Y},\mathbf{Z}])}$, (b) $\dfrac{\mathrm{ndet}([\mathbf{Q}_Z\mathbf{X},\mathbf{Q}_Z\mathbf{Y}])}{\mathrm{ndet}(\mathbf{Q}_Z\mathbf{X})\,\mathrm{ndet}(\mathbf{Q}_Z\mathbf{Y})}$,

(c) $\dfrac{\mathrm{ndet}(\mathbf{Q}_{[X,Z]}\mathbf{Y})}{\mathrm{ndet}(\mathbf{Q}_Z\mathbf{Y})}$, (d) $\dfrac{\mathrm{ndet}(\mathbf{Q}_{Q_ZX}\mathbf{Q}_Z\mathbf{Y})}{\mathrm{ndet}(\mathbf{Q}_Z\mathbf{Y})}$,

(e) $\sqrt{\det\big(\mathbf{I}_q-(\mathbf{Y}'\mathbf{Q}_Z\mathbf{Y})^{-1}(\mathbf{Y}'\mathbf{Q}_Z\mathbf{X})(\mathbf{X}'\mathbf{Q}_Z\mathbf{X})^{-1}(\mathbf{X}'\mathbf{Q}_Z\mathbf{Y})\big)}$,

(f) $\sqrt{\det\big(\mathbf{I}_n-\mathbf{P}_{Q_ZX}\mathbf{P}_{Q_ZY}\big)}$, (g) $\sqrt{\displaystyle\prod_{k=1}^{Min(p,q)}\big(1-cc_k^2(\mathbf{X},\mathbf{Y}/\mathbf{Z})\big)}$,

where $cc_k^2(\mathbf{X},\mathbf{Y}/\mathbf{Z})$ is the kth partial canonical correlation between $\mathbf{X}$ and $\mathbf{Y}$ eliminating the effect of $\mathbf{Z}$ from both $\mathbf{X}$ and $\mathbf{Y}$.

Proof. We first show the equivalence between (a) and (b). From (11) of Lemma 3, we have $\mathrm{ndet}([\mathbf{X},\mathbf{Y},\mathbf{Z}]) = \mathrm{ndet}(\mathbf{Z})\,\mathrm{ndet}\big(\mathbf{Q}_Z[\mathbf{X},\mathbf{Y}]\big) = \mathrm{ndet}(\mathbf{Z})\,\mathrm{ndet}\big([\mathbf{Q}_Z\mathbf{X},\mathbf{Q}_Z\mathbf{Y}]\big)$. Further, we have $\mathrm{ndet}([\mathbf{X},\mathbf{Z}]) = \mathrm{ndet}(\mathbf{Z})\,\mathrm{ndet}(\mathbf{Q}_Z\mathbf{X})$, and $\mathrm{ndet}([\mathbf{Y},\mathbf{Z}]) = \mathrm{ndet}(\mathbf{Z})\,\mathrm{ndet}(\mathbf{Q}_Z\mathbf{Y})$, from which (b) follows. A proof of (b) $\Rightarrow$ (a) is similarly given, using (11) of Lemma 3. To prove the equivalence between (a) and (c), one may note

$$\text{(a)} \;=\; \frac{\mathrm{ndet}([\mathbf{X},\mathbf{Y},\mathbf{Z}])/\,\mathrm{ndet}([\mathbf{X},\mathbf{Z}])}{\mathrm{ndet}([\mathbf{Y},\mathbf{Z}])/\,\mathrm{ndet}(\mathbf{Z})} \;=\; \frac{\mathrm{ndet}(\mathbf{Q}_{[X,Z]}\mathbf{Y})}{\mathrm{ndet}(\mathbf{Q}_Z\mathbf{Y})}\;.$$

To prove (c) $\Leftrightarrow$ (d), we obtain $\mathbf{Q}_{[X,Z]} = \mathbf{Q}_{Q_Z X}\mathbf{Q}_Z$, using (2) of Lemma 1. From (d) to (e), we may note

$$\begin{aligned}
&\mathrm{ndet}(\mathbf{Q}_{Q_Z X}\mathbf{Q}_Z\mathbf{Y})/\,\mathrm{ndet}(\mathbf{Q}_Z\mathbf{Y})\\
&\quad= \quad \mathrm{ndet}\big((\mathbf{I}_n - \mathbf{P}_{Q_Z X})\mathbf{Q}_Z\mathbf{Y}\big)/\,\mathrm{ndet}(\mathbf{Q}_Z\mathbf{Y})\\
&\quad= \quad \sqrt{\det\big(\mathbf{Y}'\mathbf{Q}_Z(\mathbf{I}_n - \mathbf{P}_{Q_Z X})\mathbf{Q}_Z\mathbf{Y}\big)/\det(\mathbf{Y}'\mathbf{Q}_Z\mathbf{Y})}\\
&\quad= \quad \sqrt{\det\big(\mathbf{Y}'\mathbf{Q}_Z\mathbf{Y} - \mathbf{Y}'\mathbf{Q}_Z\mathbf{X}(\mathbf{X}'\mathbf{Q}_Z\mathbf{X})^{-}\mathbf{X}'\mathbf{Q}_Z\mathbf{Y}\big)/\det(\mathbf{Y}'\mathbf{Q}_Z\mathbf{Y})},
\end{aligned}$$

establishing (e).

The equivalence among any pair of (e), (f) and (g) are easy to establish, following the definition of partial canonical correlation analysis. (See Theorem 1 of Yanai & Puntanen (1993).) (Q.E.D.)

The following corollary is a special case of Theorem 3 when all three matrices, $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$, consist of a single vector.

Corollary 6. *Let $\mathbf{X} = (\boldsymbol{x})$, $\mathbf{Y} = (\boldsymbol{y})$, and $\mathbf{Z} = (\boldsymbol{z})$. Further, let*

$$r(\boldsymbol{x},\boldsymbol{y}/\boldsymbol{z}) \;=\; r(\mathbf{Q}_z\boldsymbol{x},\mathbf{Q}_z\boldsymbol{y}) \tag{23}$$

be the partial correlation coefficient between $\boldsymbol{x}$ and $\boldsymbol{y}$ eliminating the effect of $\boldsymbol{z}$. Then, the following five quantities are equal.

$$\text{(a)}\quad \frac{\mathrm{ndet}([\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}])\,\mathrm{ndet}(\boldsymbol{z})}{\mathrm{ndet}([\boldsymbol{x},\boldsymbol{z}])\,\mathrm{ndet}([\boldsymbol{y},\boldsymbol{z}])}\;,\qquad \text{(b)}\quad \frac{\mathrm{ndet}([\mathbf{Q}_z\boldsymbol{x},\mathbf{Q}_z\boldsymbol{y}])}{\mathrm{ndet}(\mathbf{Q}_z\boldsymbol{x})\,\mathrm{ndet}(\mathbf{Q}_z\boldsymbol{y})}\;,\qquad \text{(c)}\quad \frac{\mathrm{ndet}(\mathbf{Q}_{[x,z]}\boldsymbol{y})}{\mathrm{ndet}(\mathbf{Q}_z\boldsymbol{y})}\;,$$

$$\text{(d)}\quad \frac{\mathrm{ndet}(\mathbf{Q}_{Q_z x}\mathbf{Q}_z\boldsymbol{y})}{\mathrm{ndet}(\mathbf{Q}_z\boldsymbol{y})}\;,\qquad \text{(e)}\quad \sqrt{1 - r^2(\boldsymbol{x},\boldsymbol{y}/\boldsymbol{z})}\;.$$

A direct proof of the equivalence between (a) and (e) was first given by Yanai & Ishii (2001).

Lemma 4. *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{W}$ denote matrices having the same number of rows. Then, the following six quantities are equal.*

$$\text{(a)}\quad \frac{\mathrm{ndet}([\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}])\,\mathrm{ndet}([\mathbf{Z},\mathbf{W}])}{\mathrm{ndet}([\mathbf{X},\mathbf{Z},\mathbf{W}])\,\mathrm{ndet}([\mathbf{Y},\mathbf{Z},\mathbf{W}])}\;,\qquad \text{(b)}\quad \frac{\mathrm{ndet}([\mathbf{Q}_{[Z,W]}\mathbf{X},\mathbf{Q}_{[Z,W]}\mathbf{Y}])}{\mathrm{ndet}(\mathbf{Q}_{[Z,W]}\mathbf{X})\,\mathrm{ndet}(\mathbf{Q}_{[Z,W]}\mathbf{Y})}\;,$$

$$\text{(c)}\quad \frac{\mathrm{ndet}(\mathbf{Q}_{[X,Z,W]}\mathbf{Y})}{\mathrm{ndet}(\mathbf{Q}_{[Z,W]}\mathbf{Y})}\;,\qquad \text{(d)}\quad \frac{\mathrm{ndet}([\mathbf{Q}_{Q_W Z}\mathbf{Q}_W\mathbf{X},\mathbf{Q}_{Q_W Z}\mathbf{Q}_W\mathbf{Y}])}{\mathrm{ndet}(\mathbf{Q}_{Q_W Z}\mathbf{Q}_W\mathbf{X})\,\mathrm{ndet}(\mathbf{Q}_{Q_W Z}\mathbf{Q}_W\mathbf{Y})}\;,$$

$$\text{(e)}\quad \sqrt{\prod_{j=1}^{m}\big(1 - cc_j^2(\mathbf{Q}_W\mathbf{X},\mathbf{Q}_W\mathbf{Y}/\mathbf{Q}_W\mathbf{Z})\big)}\;,\qquad \text{(f)}\quad \sqrt{\prod_{j=1}^{m}\big(1 - cc_j^2(\mathbf{X},\mathbf{Y}/[\mathbf{Z},\mathbf{W}])\big)}\;.$$

Proof. Equivalences among any pair of (a), (b), (c), and (d) can be easily proved by replacing $\mathbf{Z}$ in (a), (b), (c), and (d) of Theorem 3 by $[\mathbf{Z}, \mathbf{W}]$. (Q.E.D.)

When $\mathbf{X} = (\boldsymbol{x})$, $\mathbf{Y} = (\boldsymbol{y})$, $\mathbf{Z} = (\boldsymbol{z})$ and $\mathbf{W} = (\boldsymbol{w})$, Lemma 4 reduces to:

Corollary 7. *The following quantities are equal.*

(a) $\dfrac{\mathrm{ndet}([\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}])\,\mathrm{ndet}([\boldsymbol{z}, \boldsymbol{w}])}{\mathrm{ndet}([\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}])\,\mathrm{ndet}([\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}])}$, (b) $\dfrac{\mathrm{ndet}([\mathbf{Q}_{[z,w]}\boldsymbol{x}, \mathbf{Q}_{[z,w]}\boldsymbol{y}])}{\mathrm{ndet}(\mathbf{Q}_{[z,w]}\boldsymbol{x})\,\mathrm{ndet}(\mathbf{Q}_{[z,w]}\boldsymbol{y})}$,

(c) $\dfrac{\mathrm{ndet}(\mathbf{Q}_{[x,z,w]}\boldsymbol{y})}{\mathrm{ndet}(\mathbf{Q}_{[z,w]}\boldsymbol{y})}$, (d) $\dfrac{\mathrm{ndet}([\mathbf{Q}_{Q_w z}\mathbf{Q}_w\boldsymbol{x}, \mathbf{Q}_{Q_w z}\mathbf{Q}_w\boldsymbol{y}])}{\mathrm{ndet}(\mathbf{Q}_{Q_w z}\mathbf{Q}_w\boldsymbol{x})\,\mathrm{ndet}(\mathbf{Q}_{Q_w z}\mathbf{Q}_w\boldsymbol{y})}$,

(e) $\sqrt{1 - r^2(\mathbf{Q}_w\boldsymbol{x}, \mathbf{Q}_w\boldsymbol{y}/\mathbf{Q}_w\boldsymbol{z})}$, (f) $\sqrt{1 - r^2(\boldsymbol{x}, \boldsymbol{y}/[\boldsymbol{z}, \boldsymbol{w}])}$.

Proof. Equivalences among any pair of (a), (b), (c), and (f) can be established immediately by setting $\mathbf{X} = (\boldsymbol{x})$, $\mathbf{Y} = (\boldsymbol{y})$ and $\mathbf{Z} = [\boldsymbol{z}, \boldsymbol{w}]$ in Theorem 3. An equivalence between (d) and (e) can be shown by $\mathbf{Q}_{[x,z,w]} = \mathbf{Q}_{Q_{[x,w]}z}\mathbf{Q}_{[x,w]}$ and the equivalence between (d) and (e) of Lemma 4. We give a proof of (e) $\Leftarrow$ (f) below.

$$\begin{aligned}
\sqrt{1 - r^2(\mathbf{Q}_w\boldsymbol{x}, \mathbf{Q}_w\boldsymbol{y}/\mathbf{Q}_w\boldsymbol{z})} &= \sqrt{1 - r^2(\mathbf{Q}_{Q_w z}\mathbf{Q}_w\boldsymbol{x}, \mathbf{Q}_{Q_w z}\mathbf{Q}_w\boldsymbol{y})} \\
&= \sqrt{1 - r^2(\mathbf{Q}_{[z,w]}\boldsymbol{x}, \mathbf{Q}_{[z,w]}\boldsymbol{y})} \\
&= \sqrt{1 - r^2(\boldsymbol{x}, \boldsymbol{y}/[\boldsymbol{z}, \boldsymbol{w}])} \ . \qquad \text{(Q.E.D.)}
\end{aligned}$$

Definition 3. *Similar to Definition 2, we introduce the following definition:*

$$Re(\mathbf{A}, \mathbf{B}/\mathbf{C}) \ = \ Re(\mathbf{Q}_C\mathbf{A}, \mathbf{Q}_C\mathbf{B}). \tag{24}$$

Lemma 5.

$$Re([\mathbf{X}, \mathbf{Y}], \mathbf{Z}) \ = \ Re(\mathbf{Y}, \mathbf{Z}/\mathbf{X})Re(\mathbf{X}, \mathbf{Z}) \tag{25}$$

Proof. A proof follows, since

$$\begin{aligned}
Re([\mathbf{X}, \mathbf{Y}], \mathbf{Z}) &= \frac{\mathrm{ndet}([\mathbf{X}, \mathbf{Y}, \mathbf{Z}])}{\mathrm{ndet}([\mathbf{X}, \mathbf{Y}])\,\mathrm{ndet}(\mathbf{Z})} \\
&= \frac{\mathrm{ndet}([\mathbf{X}, \mathbf{Y}, \mathbf{Z}])\,\mathrm{ndet}(\mathbf{X})}{\mathrm{ndet}([\mathbf{X}, \mathbf{Y}])\,\mathrm{ndet}([\mathbf{X}, \mathbf{Z}])}\ \frac{\mathrm{ndet}([\mathbf{X}, \mathbf{Z}])}{\mathrm{ndet}(\mathbf{X})\,\mathrm{ndet}(\mathbf{Z})} \\
&= Re(\mathbf{Y}, \mathbf{Z}/\mathbf{X})Re(\mathbf{X}, \mathbf{Z}). \qquad \text{(Q.E.D)}
\end{aligned}$$

Using Definition 3 and the equivalence between (a) and (b) of Theorem 3, we note:

$$\begin{aligned}
Re(\mathbf{Y}, \mathbf{Z}/\mathbf{X}) &= Re(\mathbf{Q}_X\mathbf{Y}, \mathbf{Q}_X\mathbf{Z}) \\
&= \frac{\mathrm{ndet}([\mathbf{X}, \mathbf{Y}, \mathbf{Z}])\,\mathrm{ndet}(\mathbf{X})}{\mathrm{ndet}([\mathbf{X}, \mathbf{Y}])\,\mathrm{ndet}([\mathbf{X}, \mathbf{Z}])} \\
&= \frac{\mathrm{ndet}([\mathbf{X}, \mathbf{Y}, \mathbf{Z}])}{\mathrm{ndet}([\mathbf{X}, \mathbf{Y}])\,\mathrm{ndet}(\mathbf{Z})} \Bigg/ \frac{\mathrm{ndet}([\mathbf{X}, \mathbf{Z}])}{\mathrm{ndet}(\mathbf{X})\,\mathrm{ndet}(\mathbf{Z})} \\
&= Re([\mathbf{X}, \mathbf{Y}], \mathbf{Z})/Re(\mathbf{X}, \mathbf{Z}),
\end{aligned}$$

which provides an alternative proof of Lemma 5. We make the following two remarks based on Definition 3:

Note 6.

$$Re(\mathbf{Q}_W\mathbf{X}, \mathbf{Q}_W\mathbf{Y}/\mathbf{Q}_W\mathbf{Z}) \;=\; Re(\mathbf{Q}_{[Z,W]}\mathbf{X}, \mathbf{Q}_{[Z,W]}\mathbf{Y}) \;=\; Re(\mathbf{X}, \mathbf{Y}/[\mathbf{Z}, \mathbf{W}]) \;. \tag{26}$$

Proof. A proof follows immediately by substituting for $\mathbf{A} = \mathbf{Q}_W\mathbf{X}$, $\mathbf{B} = \mathbf{Q}_W\mathbf{Y}$, and $\mathbf{C} = \mathbf{Q}_W\mathbf{Z}$ in (24), that is,

$$\begin{aligned} Re(\mathbf{Q}_W\mathbf{X}, \mathbf{Q}_W\mathbf{Y}/\mathbf{Q}_W\mathbf{Z}) &= Re(\mathbf{Q}_{Q_W Z}\mathbf{Q}_W\mathbf{X}, \mathbf{Q}_{Q_W Z}\mathbf{Q}_W\mathbf{Y}) \\ &= Re(\mathbf{Q}_{[Z,W]}\mathbf{X}, \mathbf{Q}_{[Z,W]}\mathbf{Y}) \\ &= Re(\mathbf{X}, \mathbf{Y}/[\mathbf{Z}, \mathbf{W}]). \qquad \text{(Q.E.D.)} \end{aligned}$$

Note 7. Let $\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_p]$, and $\mathbf{X}_{(j)} = [\mathbf{X}_1, \ldots, \mathbf{X}_j]$. Then, using Lemma 5, $Re(\mathbf{X}_{(j)}, \mathbf{Y}) = Re([\mathbf{X}_{(j-1)}, \mathbf{X}_j], \mathbf{Y}) = Re(\mathbf{X}_j, \mathbf{Y}/\mathbf{X}_{(j-1)})Re(\mathbf{X}_{(j-1)}, \mathbf{Y})$. Repeating this formula from $j = 2$ up to p, we have

$$\begin{aligned} Re(\mathbf{X}_{(p)}, \mathbf{Y}) &= Re(\mathbf{X}_p, \mathbf{Y}/\mathbf{X}_{(p-1)})Re(\mathbf{X}_{(p-1)}, \mathbf{Y}) \\ &= Re(\mathbf{X}_p, \mathbf{Y}/\mathbf{X}_{(p-1)})Re(\mathbf{X}_{p-1}, \mathbf{Y}/\mathbf{X}_{(p-2)})Re(\mathbf{X}_{(p-2)}, \mathbf{Y}) \\ &= Re(\mathbf{X}_1, \mathbf{Y}) \prod_{j=2}^{p} Re(\mathbf{X}_j, \mathbf{Y}/\mathbf{X}_{(j-1)}) \;. \end{aligned}$$

Using Note 7, the following theorem can be established.

Theorem 4. *Let $\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_p]$ be a row block matrix with $\mathbf{X}_i$ of order $n \times m_i$ as the ith block. Let $\mathbf{Y}$ be a matrix of order $n\times$. Further, let $\mathbf{X}_{(j)} = [\mathbf{X}_1, \ldots, \mathbf{X}_j]$ be the matrix with the first j blocks of $\mathbf{X}$. Then, the following five quantities are equal, where $r \leq Min(p_1,\ p_2,\ \ldots,\ p_m)$.*

(a) $$\frac{\mathrm{ndet}([\mathbf{X}, \mathbf{Y}])}{\mathrm{ndet}(\mathbf{X})\,\mathrm{ndet}(\mathbf{Y})} \;, \tag{27}$$

(b) $$\sqrt{\prod_{k=1}^{r}\big(1 - cc_k^2(\mathbf{X}, \mathbf{Y})\big)} \;, \tag{28}$$

(c) $$\frac{\mathrm{ndet}([\mathbf{X}_1, \mathbf{Y}])}{\mathrm{ndet}(\mathbf{X}_1)\,\mathrm{ndet}(\mathbf{Y})} \prod_{j=2}^{p} \frac{\mathrm{ndet}([\mathbf{X}_{(j)}, \mathbf{Y}])\,\mathrm{ndet}(\mathbf{X}_{(j-1)})}{\mathrm{ndet}(\mathbf{X}_{(j)})\,\mathrm{ndet}([\mathbf{X}_{(j-1)}, \mathbf{Y}])} \;, \tag{29}$$

(d) $$Re(\mathbf{X}_1, \mathbf{Y})\prod_{j=2}^{p} Re(\mathbf{X}_j, \mathbf{Y}/\mathbf{X}_{(j-1)}) \;, \tag{30}$$

(e) $$\sqrt{\prod_{k=1}^{r}\big(1 - cc_k^2(\mathbf{X}_1, \mathbf{Y})\big)}\sqrt{\prod_{j=2}^{p}\prod_{k=1}^{r}\big(1 - cc_k^2(\mathbf{X}_j, \mathbf{Y}/\mathbf{X}_{(j-1)})\big)} \;. \tag{31}$$

Lemma 6. *Let $\boldsymbol{z}$ be a vector with zero mean. Further, let $mc([\mathbf{X}, \boldsymbol{y}], \boldsymbol{z})$ be the multiple correlation coefficient between $[\mathbf{X}, \boldsymbol{y}]$ and $\boldsymbol{z}$. Then, the following four quantities are equal:*

(a) $\dfrac{\mathrm{ndet}([\mathbf{X},\boldsymbol{y},\boldsymbol{z}])}{\mathrm{ndet}([\mathbf{X},\boldsymbol{y}])\,\mathrm{ndet}(\boldsymbol{z})}$, (b) $\dfrac{\mathrm{ndet}(\mathbf{Q}_{[X,y]}\boldsymbol{z})}{\mathrm{ndet}(\boldsymbol{z})}$,

(c) $\sqrt{1-mc^2([\mathbf{X},\boldsymbol{y}],\boldsymbol{z})}$, (d) $\sqrt{1-mc^2(\mathbf{X},\boldsymbol{z})}\sqrt{1-r^2(\boldsymbol{y},\boldsymbol{z}/\mathbf{X})}$.

Proof. The equivalence between (a) and (b) can easily be obtained. The proof (b) $\Rightarrow$ (d) follows from

$$\frac{\mathrm{ndet}(\mathbf{Q}_{[X,y]}\boldsymbol{z})}{\mathrm{ndet}(\boldsymbol{z})} = \frac{\mathrm{ndet}\big(\mathbf{Q}_{Q_X y}\mathbf{Q}_X\boldsymbol{z}\big)}{\mathrm{ndet}(\mathbf{Q}_X\boldsymbol{z})}\,\frac{\mathrm{ndet}(\mathbf{Q}_X\boldsymbol{z})}{\mathrm{ndet}(\boldsymbol{z})} ,$$

which follows from (2) of Lemma 1. (Q.E.D)

Corollary 8. *If both $\mathbf{X}_j$, $(j=1,\ldots,m)$, and $\mathbf{Y}$ consist of a single vector, formula (31) in Theorem 4 reduces to the following well-known formula.*

$$\sqrt{1-mc^2(\mathbf{X},\boldsymbol{y})} = \sqrt{\big(1-r^2(\boldsymbol{x}_1,\boldsymbol{y})\big)\prod_{j=2}^{p}\big(1-r^2(\boldsymbol{x}_j,\boldsymbol{y}/\mathbf{X}_{(j-1)})\big)}. \tag{32}$$

We may further decompose $Re(\mathbf{X},\mathbf{Y})$ by partitioning $\mathbf{Y}$ into $[\mathbf{Y}_1,\ldots,\mathbf{Y}_q]$.

Theorem 5. *Let $\mathbf{Y}=[\mathbf{Y}_1,\ldots,\mathbf{Y}_q]$. Then,*

$$Re(\mathbf{X},\mathbf{Y}) = \prod_{j=1}^{p}\prod_{k=1}^{q} Re(\mathbf{X}_j,\mathbf{Y}_k/[\mathbf{X}_{(j-1)},\mathbf{Y}_{(k-1)}]), \tag{33}$$

where

$$\begin{aligned}
\prod_{j=1}^{p}\prod_{k=1}^{q} Re(\mathbf{X}_j,\mathbf{Y}_k/[\mathbf{X}_{(j-1)},\mathbf{Y}_{(0)}]) &= \prod_{j=1}^{p}\prod_{k=1}^{q} Re(\mathbf{X}_j,\mathbf{Y}_k/\mathbf{X}_{(j-1)}),\\
\prod_{j=1}^{p}\prod_{k=1}^{q} Re(\mathbf{X}_j,\mathbf{Y}_k/[\mathbf{X}_{(0)},\mathbf{Y}_{(k-1)}]) &= \prod_{j=1}^{p}\prod_{k=1}^{q} Re(\mathbf{X}_j,\mathbf{Y}_k/\mathbf{Y}_{(k-1)}), \qquad \textit{and}\\
\prod_{j=1}^{p}\prod_{k=1}^{q} Re(\mathbf{X}_j,\mathbf{Y}_k/[\mathbf{X}_{(0)},\mathbf{Y}_{(0)}]) &= \prod_{j=1}^{p}\prod_{k=1}^{q} Re(\mathbf{X}_j,\mathbf{Y}_k).
\end{aligned}$$

Proof. To prove (33), observe that $Re(\mathbf{X}_j,\mathbf{Y}/\mathbf{X}_{(j-1)})$ in (30) can be decomposed as

$$\begin{aligned}
&Re(\mathbf{X}_j,\mathbf{Y}/\mathbf{X}_{(j-1)})\\
&\quad= Re(\mathbf{Q}_{X_{(j-1)}}\mathbf{X}_j,\mathbf{Q}_{X_{(j-1)}}\mathbf{Y})\\
&\quad= Re(\mathbf{Q}_{X_{(j-1)}}\mathbf{X}_j,\mathbf{Q}_{X_{(j-1)}}\mathbf{Y}_1)\big(\prod_{k=2}^{q} Re(\mathbf{Q}_{X_{(j-1)}}\mathbf{X}_j,\mathbf{Q}_{X_{(j-1)}}\mathbf{Y}_k/\mathbf{Q}_{X_{(j-1)}}\mathbf{Y}_{(k-1)})\big)\\
&\quad= \prod_{k=1}^{q} Re(\mathbf{X}_j,\mathbf{Y}_k/[\mathbf{X}_{(j-1)},\mathbf{Y}_{(k-1)}]).
\end{aligned}$$

These follow from (26).

Note 8. If $p=2$ and $q=2$, then (33) reduces to

$$Re\big([\mathbf{X}_1,\mathbf{X}_2],[\mathbf{Y}_1,\mathbf{Y}_2]\big) = Re(\mathbf{X}_2,\mathbf{Y}_2/[\mathbf{X}_1,\mathbf{Y}_1])Re(\mathbf{X}_2,\mathbf{Y}_1/\mathbf{X}_1)Re(\mathbf{X}_1,\mathbf{Y}_2/\mathbf{Y}_1)Re(\mathbf{X}_1,\mathbf{Y}_1). \tag{34}$$

4 Decompositions of the likelihood ratios

We first give the following definition of the likelihood ratios of three and four sets of variables.

Definition 4.

$$Re_{(3)}(\mathbf{X},\mathbf{Y},\mathbf{Z}) = \frac{\mathrm{ndet}([\mathbf{X},\mathbf{Y},\mathbf{Z}])}{\mathrm{ndet}(\mathbf{X})\,\mathrm{ndet}(\mathbf{Y})\,\mathrm{ndet}(\mathbf{Z})}\ , \tag{35}$$

$$Re_{(4)}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) = \frac{\mathrm{ndet}([\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}])}{\mathrm{ndet}(\mathbf{X})\,\mathrm{ndet}(\mathbf{Y})\,\mathrm{ndet}(\mathbf{Z})\,\mathrm{ndet}(\mathbf{W})}\ . \tag{36}$$

Theorem 6. *Let* $\mathbf{X},\mathbf{Y}$*, and* $\mathbf{Z}$ *denote three sets of variables with the same number of rows. Then, the following relations hold.*

$$\text{(a)}\qquad Re_{(3)}(\mathbf{X},\mathbf{Y},\mathbf{Z}) = Re\big((\mathbf{X},\mathbf{Y}),\mathbf{Z}\big)Re(\mathbf{X},\mathbf{Y})\ , \tag{37}$$

$$\text{(b)}\qquad Re_{(3)}(\mathbf{X},\mathbf{Y},\mathbf{Z}) = Re(\mathbf{X},\mathbf{Y}/\mathbf{Z})Re(\mathbf{X},\mathbf{Z})Re(\mathbf{Y},\mathbf{Z})\ . \tag{38}$$

Proof. To prove (37), we may note

$$\begin{aligned} Re_{(3)}(\mathbf{X},\mathbf{Y},\mathbf{Z}) &= \frac{\mathrm{ndet}\big([[\mathbf{X},\mathbf{Y}],\mathbf{Z}]\big)}{\mathrm{ndet}([\mathbf{X},\mathbf{Y}])\,\mathrm{ndet}(\mathbf{Z})}\ \frac{\mathrm{ndet}([\mathbf{X},\mathbf{Y}])}{\mathrm{ndet}(\mathbf{X})ndet(\mathbf{Y})} \\ &= Re([\mathbf{X},\mathbf{Y}],\mathbf{Z})Re(\mathbf{X},\mathbf{Y})\ . \end{aligned}$$

To prove (38), we may note

$$\begin{aligned} Re_{(3)}(\mathbf{X},\mathbf{Y},\mathbf{Z}) &= \frac{\mathrm{ndet}([\mathbf{X},\mathbf{Y},\mathbf{Z}])}{\mathrm{ndet}(\mathbf{X})\,\mathrm{ndet}(\mathbf{Y})\,\mathrm{ndet}(\mathbf{Z})} \\ &= \frac{\mathrm{ndet}([\mathbf{X},\mathbf{Y},\mathbf{Z}])\,\mathrm{ndet}(\mathbf{Z})}{\mathrm{ndet}([\mathbf{X},\mathbf{Z}])\,\mathrm{ndet}([\mathbf{Y},\mathbf{Z}])}\ \frac{\mathrm{ndet}([\mathbf{X},\mathbf{Z}])}{\mathrm{ndet}(\mathbf{X})\,\mathrm{ndet}(\mathbf{Z})}\ \frac{\mathrm{ndet}([\mathbf{Y},\mathbf{Z}])}{\mathrm{ndet}(\mathbf{Y})\,\mathrm{ndet}(\mathbf{Z})} \\ &= Re(\mathbf{X},\mathbf{Y}/\mathbf{Z})Re(\mathbf{X},\mathbf{Z})Re(\mathbf{Y},\mathbf{Z}). \qquad \text{(Q.E.D.)} \end{aligned}$$

Definition 5.

$$Re_{(3)}(\mathbf{X},\mathbf{Y},\mathbf{Z}/\mathbf{W}) = Re_{(3)}\big(\mathbf{Q}_W\mathbf{X},\mathbf{Q}_W\mathbf{Y},\mathbf{Q}_W\mathbf{Z}\big). \tag{39}$$

Using Definition 5 and Theorem 6, we can establish the following corollary.

Corollary 9. *Suppose we have, in addition to* $\mathbf{X}$, $\mathbf{Y}$ *and* $\mathbf{Z}$*, a fourth matrix* $\mathbf{W}$ *of the same row order as the previous three matrices. Then, it can be shown that*

$$\text{(a)}\qquad Re_{(4)}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) = Re(\mathbf{X},\mathbf{Y})Re\big([\mathbf{X},\mathbf{Y}],\mathbf{Z}\big)Re\big([\mathbf{X},\mathbf{Y},\mathbf{Z}],\mathbf{W}\big), \tag{40}$$

$$\text{(b)}\qquad Re_{(4)}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) = Re_{(3)}(\mathbf{X},\mathbf{Y},\mathbf{Z}/\mathbf{W})Re(\mathbf{X},\mathbf{W})Re(\mathbf{Y},\mathbf{W})Re(\mathbf{Z},\mathbf{W}). \tag{41}$$

Furthermore, we can show the following:

Theorem 7. *The following four quantities are equal.*

(a) $Re_{(3)}(\mathbf{X},\mathbf{Y},\mathbf{Z}/[\mathbf{W},\mathbf{V}])$,

(b) $Re_{(3)}\big(\mathbf{Q}_{[W,V]}\mathbf{X},\mathbf{Q}_{[W,V]}\mathbf{Y},\mathbf{Q}_{[W,V]}\mathbf{Z}\big)$,

(c) $Re_{(3)}\big(\mathbf{Q}_W\mathbf{X},\mathbf{Q}_W\mathbf{Y},\mathbf{Q}_W\mathbf{Z}/\mathbf{Q}_W\mathbf{V}\big)$,

(d) $Re_{(3)}\big(\mathbf{Q}_V\mathbf{X},\mathbf{Q}_V\mathbf{Y},\mathbf{Q}_V\mathbf{Z}/\mathbf{Q}_V\mathbf{W}\big)$.

Proof. A proof of the equivalence between (b) and (c), and that of (b) and (d) can be seen by noting Definition 5, and the following decompositions:

$$\mathbf{Q}_{[W,V]} = \mathbf{Q}_{Q_W V}\mathbf{Q}_W, \quad \text{and} \quad \mathbf{Q}_{[W,V]} = \mathbf{Q}_{Q_V W}\mathbf{Q}_V,$$

respectively. (Q.E.D.)

Theorem 6 and Corollary 9 can be generalized as follows:

Note 9. Given $\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_p]$, denote $\mathbf{X}_{(j)} = [\mathbf{X}_1, \ldots, \mathbf{X}_j]$, we have

$$\text{(a)} \qquad Re_{(p)}(\mathbf{X}) = \prod_{j=2}^{p} Re(\mathbf{X}_{(j-1)}, \mathbf{X}_j), \quad (p > 2), \tag{42}$$

$$\text{(b)} \qquad Re_{(p)}(\mathbf{X}) = Re_{(p-1)}(\mathbf{X}_1, \ \ldots, \ \mathbf{X}_{p-1}/\mathbf{X}_p) \prod_{j=1}^{p-1} Re(\mathbf{X}_j, \mathbf{X}_p), \tag{43}$$

$$\text{(c)} \qquad Re_{(p)}(\mathbf{X}) = 1 \ \Leftrightarrow \ \mathbf{X}_i'\mathbf{X}_j = \mathbf{O}, \quad (i \neq j). \tag{44}$$

Note 10. Formula (42) can be represented as

$$\frac{\mathrm{ndet}([\mathbf{X}_1, \ldots, \mathbf{X}_p])}{\prod_{j=1}^{p} \mathrm{ndet}(\mathbf{X}_j)} = \prod_{j=2}^{p} \frac{\mathrm{ndet}([\mathbf{X}_1, \ldots, \mathbf{X}_j])}{\mathrm{ndet}([\mathbf{X}_1, \ldots, \mathbf{X}_{j-1}])\,\mathrm{ndet}(\mathbf{X}_j)} \ . \tag{45}$$

If $\mathbf{X}_j$ $(j = 1, \ \ldots, \ p)$ are columnwise centered, it follows, from the equivalence between (a) and (f) of Theorem 2, that (45) can be written as

$$\frac{\mathrm{ndet}([\mathbf{X}_1, \ldots, \mathbf{X}_p])}{\prod_{j=2}^{p} \mathrm{ndet}(\mathbf{X}_j)} = \sqrt{\prod_{j=2}^{p} \left(1 - cc_j^2([\mathbf{X}_1, \ \ldots, \ \mathbf{X}_{j-1}], \mathbf{X}_j)\right)} \ . \tag{46}$$

Note 11. Let $\mathbf{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p]$ be an $n \times p$ columnwise centered matrix. Then, from Corollary 1, we have

$$\mathrm{ndet}(\mathbf{X}) = \mathrm{ndet}(\boldsymbol{x}_1)\,\mathrm{ndet}(\mathbf{Q}_{x_1}\boldsymbol{x}_2)\,\mathrm{ndet}(\mathbf{Q}_{[x_1,x_2]}\boldsymbol{x}_3)\cdots\mathrm{ndet}(\mathbf{Q}_{[x_1,\ldots,x_{j-1}]}\boldsymbol{x}_j)\cdots\mathrm{ndet}(\mathbf{Q}_{X_{p-1}}\boldsymbol{x}_p) \ .$$

If $\mathbf{X}$ columnwise standardized, $\mathrm{ndet}(\mathbf{X})$ reduces to

$$\mathrm{ndet}(\mathbf{X}) = \sqrt{\left(1 - r^2(\boldsymbol{x}_1, \boldsymbol{x}_2)\right)\left(1 - mc^2([\boldsymbol{x}_1, \boldsymbol{x}_2], \boldsymbol{x}_3)\right)\cdots\left(1 - mc^2(\mathbf{X}_{i-1}, \boldsymbol{x}_i)\right)\cdots\left(1 - mc^2(\mathbf{X}_{p-1}, \boldsymbol{x}_p)\right)} \tag{47}$$

where $mc^2(\mathbf{X}_{i-1}, \boldsymbol{x}_i)$ is the squared multiple correlation coefficient of $\mathbf{X}_{i-1}$ on $\boldsymbol{x}_i$.

5 Redundancy of variables in terms of the likelihood ratio

Consider a multiple regression situation, in which $\boldsymbol{y}$ is a criterion variable and $\mathbf{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2]$ is a set of predictor. It is assumed that both $\boldsymbol{y}$ and $\mathbf{X}$ are columnwise centered. In this case, the squared multiple correlation of $\boldsymbol{y}$ on $\mathbf{X}$ can be expressed, using an orthogonal projector, as $\boldsymbol{y}'\mathbf{P}_{[x_1,x_2]}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{y}$, which can be decomposed into $\boldsymbol{y}'\mathbf{P}_{x_1}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{y} + \boldsymbol{y}'\mathbf{P}_{Q_{x_1}x_2}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{y}$. If $\boldsymbol{y}'\mathbf{P}_{Q_{x_1}x_2}\boldsymbol{y} = 0$ (which is equivalent to $\boldsymbol{y}'\mathbf{Q}_{x_1}\boldsymbol{x}_2 = 0$), then $mc^2([\boldsymbol{x}_1, \boldsymbol{x}_2], \boldsymbol{y}) = r^2(\boldsymbol{x}_1, \boldsymbol{y})$, which implies that adding $\boldsymbol{x}_2$ to $\boldsymbol{x}_1$ does not provide any new information, that is, $\boldsymbol{x}_2$ is completely redundant with $\boldsymbol{x}_1$. In this section, we consider the redundancy of variables defined in terms of the likelihood ratios of two and three sets of variables, using the nonnegative determinants of a rectangular matrix.

Theorem 8. *Let* $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2]$ *and* $\mathbf{Y} = [\mathbf{Y}_1, \mathbf{Y}_2]$*, each partitioned into two sets. Then, the following relationship,*

$$Re(\mathbf{X}_1, \mathbf{Y}_1) \geq Re(\mathbf{X}, \mathbf{Y}) \tag{48}$$

holds. The equality in holds if the following three conditions,

$$\text{(i)} \quad \mathbf{X}_2'\mathbf{Q}_{X_1}\mathbf{Y}_1 = \mathbf{O}, \qquad \text{(ii)} \quad \mathbf{X}_1'\mathbf{Q}_{Y_1}\mathbf{Y}_2 = \mathbf{O}, \qquad \text{and} \qquad \text{(iii)} \quad \mathbf{X}_2'\mathbf{Q}_{[X_1,Y_1]}\mathbf{Y}_2 = \mathbf{O} \tag{49}$$

hold simultaneously.

Proof. A proof of (48) can be easily obtained given by (34), noting that $Re(\mathbf{X}_2, \mathbf{Y}_2/[\mathbf{X}_1, \mathbf{Y}_1])$, $Re(\mathbf{X}_2, \mathbf{Y}_1/\mathbf{X}_1)$, and $Re(\mathbf{X}_1, \mathbf{Y}_2/\mathbf{Y}_1)$ are all strictly less than 1. Condition (49) holds when

$$Re(\mathbf{X}_2, \mathbf{Y}_2/[\mathbf{X}_1, \mathbf{Y}_1]) = 1, \quad Re(\mathbf{X}_2, \mathbf{Y}_1/\mathbf{X}_1) = 1, \quad \text{and} \quad Re(\mathbf{X}_1, \mathbf{Y}_2/\mathbf{Y}_1) = 1$$

hold simultaneously. (Q.E.D.)

Note 12. Condition (iii) of (49) may be replaced by either (iv) $\mathbf{X}_2'\mathbf{Q}_{X_1}\mathbf{Y}_2 = \mathbf{O}$ or (v) $\mathbf{X}_2'\mathbf{Q}_{Y_1}\mathbf{Y}_2 = \mathbf{O}$.

When $\mathbf{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2]$ and $\mathbf{Y} = (\boldsymbol{y})$ as a special case, (48) becomes

$$Re(\boldsymbol{x}_1, \boldsymbol{y}) \;\geq\; Re\big([\boldsymbol{x}_1, \boldsymbol{x}_2], \boldsymbol{y}\big)\ . \tag{50}$$

The equality in (50) holds when $Re(\boldsymbol{x}_2, \boldsymbol{y}/\boldsymbol{x}_1) = 1$, which implies $\boldsymbol{y}'\mathbf{Q}_{x_1}\boldsymbol{x}_2 = 0$, and consequently, $r(\boldsymbol{x}_2, \boldsymbol{y}/\boldsymbol{x}_1) = 0$. Thus, the condition required for $\boldsymbol{x}_2$ to be completely redundant with $\boldsymbol{x}_1$ is exactly the same as the condition introduced for the equality of the squared multiple correlation coefficients.

If $\mathbf{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2]$ and $\mathbf{Y} = [\boldsymbol{y}_1, \boldsymbol{y}_2]$, then from (34),

$$\begin{aligned} Re(\mathbf{X}, \mathbf{Y}) &= \frac{\operatorname{ndet}([\mathbf{X}, \mathbf{Y}])}{\operatorname{ndet}(\mathbf{X})\operatorname{ndet}(\mathbf{Y})} \\ &= Re\big([\boldsymbol{x}_1, \boldsymbol{x}_2], [\boldsymbol{y}_1, \boldsymbol{y}_2]\big) \\ &= Re(\boldsymbol{x}_2, \boldsymbol{y}_2/[\boldsymbol{x}_1, \boldsymbol{y}_1])Re(\boldsymbol{x}_2, \boldsymbol{y}_1/\boldsymbol{x}_1)Re(\boldsymbol{x}_1, \boldsymbol{y}_2/\boldsymbol{y}_1)Re(\boldsymbol{x}_1, \boldsymbol{y}_1). \end{aligned} \tag{51}$$

Thus, $Re(\boldsymbol{x}_1, \boldsymbol{y}_1) \geq Re\big([\boldsymbol{x}_1, \boldsymbol{x}_2], [\boldsymbol{y}_1, \boldsymbol{y}_2]\big)$, and the equality holds when $Re(\boldsymbol{x}_2, \boldsymbol{y}_2/\boldsymbol{x}_1, \boldsymbol{y}_1) = Re(\boldsymbol{x}_2, \boldsymbol{y}_1/\boldsymbol{x}_1) = Re(\boldsymbol{x}_1, \boldsymbol{y}_2/\boldsymbol{y}_1) = 1$, which implies that $\boldsymbol{x}_2'\mathbf{Q}_{[x_1,y_1]}\boldsymbol{y}_2 = 0$, $\boldsymbol{x}_2'\mathbf{Q}_{x_1}\boldsymbol{y}_1 = 0$, and $\boldsymbol{x}_1'\mathbf{Q}_{y_1}\boldsymbol{y}_2 = 0$ hold simultaneously. When these three equations hold simultaneously, we obtain

$$Re\big([\boldsymbol{x}_1, \boldsymbol{x}_2], [\boldsymbol{y}_1, \boldsymbol{y}_2]\big) \;=\; Re(\boldsymbol{x}_1, \boldsymbol{y}_1),$$

which implies that $\boldsymbol{x}_2$ and $\boldsymbol{y}_2$ are redundant with $\boldsymbol{x}_1$ and $\boldsymbol{y}_1$, respectively.

In closing, we consider the redundancy of variables when given variables are partitioned into three sets.

Note 13. Let $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2]$, $\mathbf{Y} = [\mathbf{Y}_1, \mathbf{Y}_2]$, and $\mathbf{Z} = [\mathbf{Z}_1, \mathbf{Z}_2]$. Assume further that $Re_{(3)}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \neq 0$. Then, $Re_{(3)}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = Re_{(3)}(\mathbf{X}_1, \mathbf{Y}_1, \mathbf{Z}_1)$ implies both

$$\text{(a)} \quad Re\big([\mathbf{X}, \mathbf{Y}], \mathbf{Z}\big) = Re\big([\mathbf{X}_1, \mathbf{Y}_1], \mathbf{Z}_1\big), \qquad \text{and} \qquad \text{(b)} \quad Re(\mathbf{X}, \mathbf{Y}) = Re(\mathbf{X}_1, \mathbf{Y}_1).$$

Furthermore, in order for both (a) and (b) to hold, the following three conditions,

$$(1) \quad \begin{bmatrix} \mathbf{X}_2' \\ \mathbf{Y}_2' \end{bmatrix} \mathbf{Q}_{[X_1,Y_1]}\mathbf{Z}_1 = \mathbf{O} \ , \qquad (2) \quad \begin{bmatrix} \mathbf{X}_1' \\ \mathbf{Y}_1' \end{bmatrix} \mathbf{Q}_{Z_1}\mathbf{Z}_2 = \mathbf{O},$$

and

$$(3) \quad \begin{bmatrix} \mathbf{X}_2' \\ \mathbf{Y}_2' \end{bmatrix} \mathbf{Q}_{[X_1,Y_1,Z_1]}\mathbf{Z}_2 = \mathbf{O},$$

must hold simultaneously. Note that condition (3) above may be replaced by either one of the following conditions:

$$(4) \quad \begin{bmatrix} \mathbf{X}_2' \\ \mathbf{Y}_2' \end{bmatrix} \mathbf{Q}_{[X_1,Y_1]}\mathbf{Z}_2 = \mathbf{O} \ , \qquad (5) \quad \begin{bmatrix} \mathbf{X}_2' \\ \mathbf{Y}_2' \end{bmatrix} \mathbf{Q}_{Z_1}\mathbf{Z}_2 = \mathbf{O}.$$

A proof of Note 13 follows immediately by observing that $Re(\mathbf{X}_1, \mathbf{Y}_1, \mathbf{Z}_1) = Re([\mathbf{X}_1, \mathbf{Y}_1], \mathbf{Z}_1)Re(\mathbf{X}_1, \mathbf{Y}_1) \geq Re([\mathbf{X}, \mathbf{Y}], \mathbf{Z})Re(\mathbf{X}_1, \mathbf{Y}_1) \geq Re([\mathbf{X}, \mathbf{Y}], \mathbf{Z})Re(\mathbf{X}, \mathbf{Y}) = Re_{(3)}(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ from (37) and (48). Conditions (1) through (5) above can easily be established, following a similar line of proof for condition (49).

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