SCHUR COMPLEMENTS AND BANACHIEWICZ-SCHUR FORMS*

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Abstract. Through the matrix rank method, this paper gives necessary and sufficient conditions for a partitioned matrix to have generalized inverses with Banachiewicz-Schur forms. In addition, this paper investigates the idempotency of generalized Schur complements in a partitioned idempotent matrix.

Key words. Banachiewicz-Schur form, Generalized inverse, Generalized Schur complement, Idempotent matrix, Matrix rank method, Maximal rank, Minimal rank, Moore-Penrose inverse, Partitioned matrix.

AMS subject classifications. 15A03, 15A09.

1. Introduction. Let $\mathbb{C}^{m\times n}$ denote the set of all $m\times n$ matrices over the field of complex numbers. The symbols A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, the rank and the range (column space) of a matrix $A\in\mathbb{C}^{m\times n}$, respectively; $[\,A,\, B\,]$ denotes a row block matrix consisting of A and B.

A matrix $X\in\mathbb{C}^{n\times m}$ is called a g-inverse of $A\in\mathbb{C}^{m\times n}$, denoted by $X = A^-$, if they satisfy $AXA = A$. The collection of all A^- is denoted by $A\{1\}$. When A is nonsingular, $A^- = A^{-1}$. In addition to A^-, there are also other g-inverses satisfying some additional equations. The well-known Moore-Penrose inverse $A^\dagger$ of $A\in\mathbb{C}^{m\times n}$ is defined to be the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying the following four Penrose equations

(i) $AXA = A$,
(ii) $XAX = X$,
(iii) $(AX)^* = AX$,
(iv) $(XA)^* = XA$.

An X is called an $\{i,\ldots,j\}$-inverse of A, denoted by $A^{(i,\ldots,j)}$, if it satisfies the $i,\ldots,j$th equations, while the collection of all $\{i,\ldots,j\}$-inverses of A is denoted by $A\{i,\ldots,j\}$. In particular, $\{1,\,2\}$-inverse of A is called reflexive g-inverse of A; $\{1,\,3\}$-inverse of A is called least squares g-inverse of A; $\{1,\,4\}$-inverse of A is called minimum norm g-inverse of A. For simplicity, let $P_A = I_m - AA^-$ and $Q_A = I_n - A^-A$.

Generalized inverses of block matrices have been a main concern in the theory of generalized inverses and applications. Let M be a 2×2 block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.1}$$

where $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{m\times k}$, $C\in\mathbb{C}^{l\times n}$ and $D\in\mathbb{C}^{l\times k}$. The generalized Schur

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complement of A in M is defined by

$$S = D - CA^{-}B. \tag{1.2}$$

If both M and A in (1.1) are nonsingular, then $S = D - CA^{-1}B$ is nonsingular, too, and M can be decomposed as

$$M = \begin{bmatrix} I_m & 0 \\ CA^{-1} & I_l \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I_m & A^{-1}B \\ 0 & I_l \end{bmatrix}, \tag{1.3}$$

where I_t is the identity matrix of order t. In this case, the inverse of M can be written as

$$\begin{aligned} M^{-1} &= \begin{bmatrix} I_m & -A^{-1}B \\ 0 & I_l \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^{-1} & I_l \end{bmatrix} \\ &= \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}. \end{aligned} \tag{1.4}$$

Result (1.4) is well known and has extensively been used in dealing with inverses of block matrices.

Motivated by (1.4), the Banachiewicz-Schur form of M in (1.1) is defined by

$$\begin{aligned} N(A^{-}) &= \begin{bmatrix} I_n & -A^{-}B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A^{-} & 0 \\ 0 & S^{-} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^{-} & I_l \end{bmatrix} \\ &= \begin{bmatrix} A^{-} + A^{-}BS^{-}CA^{-} & -A^{-}BS^{-} \\ -S^{-}CA^{-} & S^{-} \end{bmatrix}, \end{aligned} \tag{1.5}$$

where S is defined as in (1.2). The matrix $N(A^{-})$ in (1.5) may vary with respect to the choice of A^{-} and S^{-}. In this case, the collection of all $N(A^{-})$ is denoted by $\{N(A^{-})\}$. It should be pointed out that $N(A^{-})$ is not necessarily a g-inverse of M for some given A^{-} and S^{-} although $N(A^{-})$ is an extension of (1.4). Many authors have investigated the relations between M^{-} and $N(A^{-})$, see, e.g., [1, 3, 5, 11] among others. Because M^{-}, A^{-} and S^{-} can be taken as $\{i, \ldots, j\}$-inverses, the equality $M^{-} = N(A^{-})$ has a variety of different expressions. In these cases, it is of interest to give necessary and sufficient conditions for each equality to hold. In this paper, we shall establish a variety of formulas for the ranks of the difference

$$M^{-} - N(A^{-}), \tag{1.6}$$

and then use the rank formulas to characterize the corresponding equality.

Some useful rank formulas for partitioned matrices and g-inverses are given in the following lemma.

LEMMA 1.1 ([10]). *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ *and* $D \in \mathbb{C}^{l\times k}$. *Then:*

(a) $r[\,A,\ B\,] = r(A) + r(\,B - AA^{-}B\,) = r(B) + r(\,A - BB^{-}A\,)$.

(b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(\,C - CA^{-}A\,) = r(C) + r(\,A - AC^{-}C\,)$.

(c) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r[\,(\,I_m - BB^-\,)A(\,I_n - C^-C\,)\,].$

(d) $r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & B - AA^-B \\ C - CA^-A & D - CA^-B \end{bmatrix}.$

The formulas in Lemma 1.1 can be used to simplify various matrix expressions involving g-inverses. For example,

$$r\begin{bmatrix} P_{B_1}A_1 \\ P_{B_2}A_2 \end{bmatrix} = r\begin{bmatrix} B_1 & 0 & A_1 \\ 0 & B_2 & A_2 \end{bmatrix} - r(B_1) - r(B_2), \tag{1.7}$$

$$r[\,D_1Q_{C_1},\, D_2Q_{C_2}\,] = r\begin{bmatrix} D_1 & D_2 \\ C_1 & 0 \\ 0 & C_2 \end{bmatrix} - r(C_1) - r(C_2), \tag{1.8}$$

$$r\begin{bmatrix} A & BQ_{B_1} \\ P_{C_1}C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & C_1 \\ 0 & B_1 & 0 \end{bmatrix} - r(B_1) - r(C_1), \tag{1.9}$$

(1.10)

$$r\begin{bmatrix} P_{B_1}AQ_{C_1} & P_{B_1}BQ_{B_2} \\ P_{C_2}CQ_{C_1} & 0 \end{bmatrix} = r\begin{bmatrix} A & B & B_1 & 0 \\ C & 0 & 0 & C_2 \\ C_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \end{bmatrix} - r(B_1) - r(B_2) - r(C_1) - r(C_2).$$

The generalized Schur complement $D - CA^-B$ in (1.2) may vary with respect to the choice of A^-. In this case, the following results give the maximal and minimal ranks of $D - CA^-B$ and its properties.

Lemma 1.2 ([15]). *Let M and S be as given in* (1.1) *and* (1.2). *Then*

$$\max_{A^-} r(\,D - CA^-B\,) = \min\left\{ r[\,C,\, D\,],\quad r\begin{bmatrix} B \\ D \end{bmatrix},\quad r(M) - r(A) \right\}, \tag{1.11}$$

$$\min_{A^-} r(\,D - CA^-B\,) = r(A) + r(M) + r[\,C,\, D\,] + r\begin{bmatrix} B \\ D \end{bmatrix} - \begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} \tag{1.12}$$
$$- r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix},$$

$$\max_{A^{(1,2)}} r(\,D - CA^{(1,2)}B\,) = \min\left\{ r(A) + r(D),\, r[\,C,\, D\,],\, r\begin{bmatrix} B \\ D \end{bmatrix},\, r(M) - r(A) \right\}, \tag{1.13}$$

$$\max_{A^{(1,3)}} r(\,D - CA^{(1,3)}B\,) = \min\left\{ r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} - r(A),\quad r\begin{bmatrix} B \\ D \end{bmatrix} \right\}, \tag{1.14}$$

$$\max_{A^{(1,4)}} r(\,D - CA^{(1,4)}B\,) = \min\left\{ r[\,C,\, D\,],\; r\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} - r(A) \right\}. \tag{1.15}$$

Corollary 1.3. *Let M be as given in* (1.1). *Then there is A^- such that*

$$r(M) = r(A) + r(\,D - CA^-B\,) \tag{1.16}$$

if and only if

$$r(M) \leqslant r(A) + \min\left\{ r[\,C,\, D\,], \quad r\begin{bmatrix} B \\ D \end{bmatrix} \right\}. \tag{1.17}$$

Equality (1.16) *holds for any* A^- *if and only if*

$$r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} = r(A) + r[\,C\ D\,] \text{ and } r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} B \\ D \end{bmatrix}, \tag{1.18}$$

that is, $\mathscr{R}[\,(P_AB)^*] \subseteq \mathscr{R}[\,(P_CD)^*]$ *and* $\mathscr{R}(CQ_A) \subseteq \mathscr{R}(DQ_B)$.

Proof. Note from (1.11) that $r(M) - r(A)$ is an upper bound for $r(\,D - CA^-B\,)$. Thus, there is A^- such that (1.16) holds if and only if

$$\max_{A^-} r(\,D - CA^-B\,) = r(M) - r(A).$$

Substituting (1.11) into this and simplifying yield (1.17). Equality (1.16) holds for any A^- if and only if

$$\min_{A^-} r(\,D - CA^-B\,) = r(M) - r(A).$$

Substituting (1.12) into this yields

$$\left(r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - r[\,C,\, D\,] - r(A) \right) + \left(r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix} - r\begin{bmatrix} B \\ D \end{bmatrix} - r(A) \right) = 0.$$

Note that both terms are nonnegative. Hence this equality is equivalent to (1.18). Applying Lemma 1.1(a) and (b) to the block matrices in (1.18) yields

$$r\begin{bmatrix} P_AB \\ P_CD \end{bmatrix} = r(P_CD) \quad \text{and} \quad r[\,CQ_A,\, DQ_B\,] = r(DQ_B).$$

These two rank equalities are obviously equivalent to $\mathscr{R}[\,(P_AB)^*] \subseteq \mathscr{R}[\,(P_CD)^*]$ and $\mathscr{R}(CQ_A) \subseteq \mathscr{R}(DQ_B)$. □

Equality (1.16) is in fact a rank equation for the partitioned matrix M. For given A, A^-, B and C, it is of interest to find D such that (1.16) holds. Other two basic rank equations for the partitioned matrix M are given by

$$r\begin{bmatrix} A & B \\ C & X \end{bmatrix} = r(A), \quad r\begin{bmatrix} A & B \\ C & X \end{bmatrix} = r\begin{bmatrix} A & B \\ C & CA^-B \end{bmatrix} + r(\,X - CA^-B\,).$$

The first one has been investigated in [9, 13].

Lemma 1.4 ([15]). *Let* $A \in \mathbb{C}^{m\times n}$ *and* $G \in \mathbb{C}^{n\times m}$. *Then*

$$\min_{A^-} r(\,A^- - G\,) = r(\,A - AGA\,), \tag{1.19}$$

$$\min_{A^{(1,2)}} r(\,A^{(1,2)} - G\,) = \max\{\,r(\,A - AGA\,),\ r(G) + r(A) - r(GA) - r(AG)\,\}, \tag{1.20}$$

$$\min_{A^{(1,3)}} r(\,A^{(1,3)} - G\,) = r(\,A^*AG - A^*\,), \tag{1.21}$$

$$\min_{A^{(1,4)}} r(\,A^{(1,4)} - G\,) = r(\,GAA^* - A^*\,). \tag{1.22}$$

When A and G are some given block matrices, the rank formulas on the right-hand sides of (1.19)–(1.22) can possibly be simplified by elementary matrix operations. In these cases, some necessary and sufficient conditions for G to be $\{1\}$-, $\{1,2\}$-, $\{1,3\}$- and $\{1,4\}$-inverses of A can be derived from the rank formulas. We shall use this rank method to establish necessary and sufficient conditions for $N(A^-)$ in (1.5) to be $\{1\}$-, $\{1,2\}$-, $\{1,3\}$- and $\{1,4\}$-inverses of M in (1.1).

2. Generalized inverses of partitioned matrices. In this section, we show a group of formulas for the rank of the difference in (1.6) and then use the formulas to characterize the equality $M^- = N(A^-)$.

THEOREM 2.1. *Let $N(A^-)$ be as given in* (1.5). *Then*

$$r[\, N(A^-)\,] = r(A^-) + r[\,(\, D - CA^-B\,)^-\,]. \tag{2.1}$$

Proof. It follows from the first equality in (1.5). $\square$

THEOREM 2.2. *Let M and $N(A^-)$ be as given in* (1.1) *and* (1.5), *respectively. Then*

$$\min_{M^-} r[\, M^- - N(A^-)\,] = r(M) - r(A) - r(\, D - CA^-B\,). \tag{2.2}$$

Hence, the following statements are equivalent:

(a) $N(A^-) \in M\{1\}$.

(b) *The g-inverse A^- in $N(A^-)$ satisfies $r(M) = r(A) + r(\, D - CA^-B\,)$.*

(c) *The g-inverse A^- in $N(A^-)$ satisfies*

$$r\begin{bmatrix} 0 & B - AA^-B \\ C - CA^-A & D - CA^-B \end{bmatrix} = r(\, D - CA^-B\,).$$

Proof. It follows from (1.19) that

$$\min_{M^-} r[\, M^- - N(A^-)\,] = r[\, M - MN(A^-)M\,]. \tag{2.3}$$

It is easy to verify that

$$M - MN(A^-)M = \begin{bmatrix} -P_ABS^-CQ_A & P_ABQ_S \\ P_SCQ_A & 0 \end{bmatrix},$$

where $S = D - CA^-B$. Recall that elementary block matrix operations do not change the rank of matrix. Applying (1.10) and elementary block matrix operations to the matrix on the right-hand side leads to

$$\begin{aligned} r[\, M - MN(A^-)M\,] &= r\begin{bmatrix} P_ABS^-CQ_A & P_ABQ_S \\ P_SCQ_A & 0 \end{bmatrix} \\ &= r\begin{bmatrix} BS^-C & B & A & 0 \\ C & 0 & 0 & S \\ A & 0 & 0 & 0 \\ 0 & S & 0 & 0 \end{bmatrix} - 2r(A) - 2r(S) \end{aligned}$$

$$
\begin{aligned}
&= r\begin{bmatrix} 0 & B & A & 0 \\ C & 0 & 0 & S \\ A & 0 & 0 & 0 \\ -SS^{-}C & S & 0 & 0 \end{bmatrix} - 2r(A) - 2r(S) \\
&= r\begin{bmatrix} 0 & B & A & 0 \\ C & 0 & 0 & S \\ A & 0 & 0 & 0 \\ 0 & S & 0 & S \end{bmatrix} - 2r(A) - 2r(S) \\
&= r\begin{bmatrix} 0 & B & A \\ C & -S & 0 \\ A & 0 & 0 \end{bmatrix} - 2r(A) - r(S) \\
&= r\begin{bmatrix} 0 & B & A \\ C & -D + CA^{-}B & 0 \\ A & 0 & 0 \end{bmatrix} - 2r(A) - r(S) \\
&= r\begin{bmatrix} 0 & B & A \\ C & -D & 0 \\ A & 0 & A \end{bmatrix} - 2r(A) - r(S) \\
&= r\begin{bmatrix} -A & B & 0 \\ C & -D & 0 \\ 0 & 0 & A \end{bmatrix} - 2r(A) - r(S) \\
&= r(M) - r(A) - r(S).
\end{aligned}
$$

Substituting this into (2.3) gives (2.2). The equivalence of (a) and (b) follows from (2.2). The equivalence of (b) and (c) follows from Lemma 1.1(d). □

Theorem 2.2 gives necessary and sufficient conditions for a given $N(A^{-})$ to be a generalized inverse of M. From Theorem 2.2, we are also able to give the existence of A^{-} so that $N(A^{-}) \in M\{1\}$.

THEOREM 2.3. *Let M and $N(A^{-})$ be as given in* (1.1) *and* (1.5), *respectively. Then*:

(a) *The following statements are equivalent*:
  (i) *There is A^{-} such that $N(A^{-}) \in M\{1\}$.*
  (ii) *There is A^{-} such that $r(M) = r(A) + r(D - CA^{-}B)$.*
  (iii) *The rank of M satisfies the following inequality*

$$
r(M) \leqslant \min\left\{ r(A) + r\begin{bmatrix} B \\ D \end{bmatrix}, \quad r(A) + r[\,C,\, D\,] \right\}. \tag{2.4}
$$

(b) *The following statements are equivalent*:
  (i) *The set inclusion $\{N(A^{-})\} \subseteq M\{1\}$ holds.*
  (ii) *$r(M) = r(A) + r(D - CA^{-}B)$ holds for any A^{-}.*
  (iii) (1.18) *holds.*

Proof. Substituting (1.11) and (1.12) into (2.2) gives

$$
\min_{A^{-},\, M^{-}} r[\, M^{-} - N(A^{-})\,]
$$

$$= \max\left\{ r(M) - r(A) - r[\,C,\, D\,],\quad r(M) - r(A) - r\begin{bmatrix} B \\ D \end{bmatrix},\quad 0 \right\},$$

$$\max_{A^-}\min_{M^-} r[\,M^- - N(A^-)\,]$$

$$= \left(r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - r[\,C,\, D\,] - r(A) \right) + \left(r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix} - r\begin{bmatrix} B \\ D \end{bmatrix} - r(A) \right).$$

Combining these two equalities with Corollary 1.3 results in the equivalences in (a) and (b). □

Theorem 2.4. *Let M and $N(A^-)$ be as given in* (1.1) *and* (1.5), *respectively. Then the following statements are equivalent*:

(a) $N(A^-) \in M\{1,2\}$.

(b) *The g-inverses A^- and S^- in $N(A^-)$ satisfy $A^- \in A\{1,2\}$, $S^- \in S\{1,2\}$ and $r(M) = r(A) + r(\,D - CA^-B\,)$.*

Proof. Recall that $X \in A\{1,2\}$ if and only if $X \in A\{1\}$ and $r(X) = r(A)$. Hence, $N(A^-) \in M\{1,2\}$ if and only if $N(A^-) \in M\{1\}$ and $r[\,N(A^-)\,] = r(M)$. From Theorem 2.2(a) and (b), $N(A^-) \in M\{1\}$ is equivalent to

$$r(M) = r(A) + r(\,D - CA^-B\,). \tag{2.5}$$

From Theorem 2.1, $r[\,N(A^-)\,] = r(M)$ is equivalent to

$$r(M) = r(A^-) + r[\,(\,D - CA^-B\,)^-\,]. \tag{2.6}$$

Also note that $r(A^-) \geqslant r(A)$ and $r[\,(\,D - CA^-B\,)^-\,] \geqslant r(\,D - CA^-B\,)$. Hence, (2.5) and (2.6) imply that $r(A^-) = r(A)$ and $r[\,(\,D - CA^-B\,)^-\,] = r(\,D - CA^-B\,)$. These two rank equalities show that $A^- \in A\{1,2\}$ and $S^- \in S\{1,2\}$. Thus (a) implies (b). Conversely, the third equality in (b) implies $N(A^-) \in M\{1\}$ by Theorem 2.2(a) and (b). If $A^- \in A\{1,2\}$ and $S^- \in S\{1,2\}$, then we see from Theorem 2.1 that

$$r[\,N(A^-)\,] = r(A^{(1,2)}) + r(S^{(1,2)}) = r(A) + r(S) = r(M).$$

Thus $N(A^-) \in M\{1,2\}$. □

Theorem 2.5. *Let M and $N(A^-)$ be as given in* (1.1) *and* (1.5), *respectively. Then there are $A^- \in A\{1,2\}$ and $S^- \in S\{1,2\}$ such that $N(A^-) \in M\{1,2\}$ if and only if*

$$r(M) \leqslant \min\left\{ 2r(A) + r(D),\quad r(A) + r\begin{bmatrix} B \\ D \end{bmatrix},\quad r(A) + r[\,C,\, D\,] \right\} \tag{2.7}$$

holds.

Proof. It follows from (1.20) that

$$\min_{M^- \in M\{1,2\}} r[\,M^- - N(A^-)\,] = \max\{\, r_1,\quad r_2\,\}, \tag{2.8}$$

where

$$r_1 = r[\,M - MN(A^-)M\,],\quad r_2 = r(M) + r[\,N(A^-)\,] - r[\,N(A^-)M\,] - r[\,MN(A^-)\,].$$

When $A^- \in A\{1,2\}$ and $S^- \in S\{1,2\}$,

$$r[\,M - MN(A^-)M\,] = r(M) - r(A) - r(S)$$

holds by (2.2). It is easy to verify that

$$MN(A^-) = \begin{bmatrix} AA^- & P_A BS^- \\ CA^- & SS^- \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^- & I_l \end{bmatrix}, \tag{2.9}$$

$$N(A^-)M = \begin{bmatrix} I_n & -A^-B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A^-A & A^-B \\ S^-CQ_A & S^-S \end{bmatrix}. \tag{2.10}$$

When $A^- \in A\{1,2\}$ and $S^- \in S\{1,2\}$, we can find by (2.1) and elementary block matrix operations that

$$r[\,N(A^-)\,] = r[\,MN(A^-)\,] = r[\,N(A^-)M\,] = r(A) + r(S).$$

Hence,

$$\min_{M^- \in M\{1,2\}} r[\,M^- - N(A^-)\,] = r(M) - r(A) - r(\,D - CA^-B\,). \tag{2.11}$$

Substituting (1.13) into (2.11) gives

$$\min_{M^- \in M\{1,2\}, A^- \in A\{1,2\}} r[\,M^- - N(A^-)\,] = \max\{r_1, \quad r_2, \quad r_3, \quad 0\}, \tag{2.12}$$

where

$$r_1 = r(M) - 2r(A) - r(D), \; r_2 = r(M) - r(A) - r[\,C,\, D\,], \; r_3 = r(M) - r(A) - r\begin{bmatrix} B \\ D \end{bmatrix}.$$

Let the right-hand side of (2.12) be zero, we obtain (2.7). □

Theorem 2.6. *Let M be given in* (1.1) *and $N(A^-)$ in* (1.5). *Then*

$$\min_{M^- \in M\{1,3\}} r[\,M^- - N(A^-)\,] = r\begin{bmatrix} A^*P_A & C^*P_S \\ B^*P_A & D^*P_S \end{bmatrix}. \tag{2.13}$$

Hence the following statements are equivalent:

(a) $N(A^-) \in M\{1,3\}$.

(b) *The g-inverses A^- and S^- in $N(A^-)$ satisfy $A^*P_A = 0$, $C^*P_S = 0$, $B^*P_A = 0$ and $D^*P_S = 0$.*

(c) *$\mathscr{R}(B) \subseteq \mathscr{R}(A)$, $\mathscr{R}(C) \subseteq \mathscr{R}(S)$ and the g-inverses A^- and S^- in $N(A^-)$ satisfy $A^- \in A\{1,3\}$ and $S^- \in S\{1,3\}$.*

Proof. From (1.21)

$$\min_{M^- \in M\{1,3\}} r[\,M^- - N(A^-)\,] = r[\,M^*MN(A^-) - M^*\,].$$

From (2.9)

$$\begin{aligned} M^*MN(A^-) - M^* &= -M^*[\,I_{m+l} - MN(A^-)\,] \\ &= -M^*\begin{bmatrix} P_A + P_ABS^-CA^- & -P_ABS^- \\ -P_SCA^- & P_S \end{bmatrix}. \end{aligned}$$

Also by elementary block matrix operations

$$r[\, M^*MN(A^-) - M^* \,] = r\left(M^* \begin{bmatrix} P_A + P_ABS^-CA^- & -P_ABS^- \\ -P_SCA^- & P_S \end{bmatrix} \right)$$
$$= r\left(M^* \begin{bmatrix} P_A & 0 \\ 0 & P_S \end{bmatrix} \right).$$

Thus we have (2.13). The equivalence of (a) and (b) is derived from (2.13). Note $A^*P_A = 0$ is $A^*AA^- = A^*$. This is equivalent to $A^- \in A\{1,3\}$, this is, $AA^- = AA^\dagger$. In this case, $B^*P_A = 0$ is $B^*AA^\dagger = A^*$, which is equivalent to $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. The two equalities $C^*P_S = 0$ and $D^*P_S = 0$ imply that $(\, D - CA^-B \,)^*P_S = S^* - S^*SS^- = 0$, which is equivalent to $S^- \in S\{1,3\}$. In this case, $C^*P_S = 0$ is equivalent to $\mathscr{R}(C) \subseteq \mathscr{R}(S)$. Conversely, if (c) holds, then the first three equalities in (b) follow immediately. Also note that $\mathscr{R}[\, C,\, S\,] = \mathscr{R}[\, C,\, D\,]$ and that $\mathscr{R}(C) \subseteq \mathscr{R}(S)$ implies $\mathscr{R}[\, C,\, S\,] = \mathscr{R}(S)$. Hence $\mathscr{R}[\, C,\, D\,] = \mathscr{R}(S)$. This shows that $\mathscr{R}(D) \subseteq \mathscr{R}(S)$. Thus $D^*P_S = 0$. □

THEOREM 2.7. *Let M and $N(A^-)$ be as given in* (1.1) *and* (1.5), *respectively. Then there are $A^- \in A\{1,3\}$ and $S^- \in S\{1,3\}$ such that $N(A^-) \in M\{1,3\}$ if and only if*

$$\mathscr{R}(B) \subseteq \mathscr{R}(A),\ \mathscr{R}\begin{bmatrix} A^*A \\ B^*A \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} C^* \\ D^* \end{bmatrix} = \{0\} \text{ and } r[\, C,\, D\,] \leqslant r\begin{bmatrix} B \\ D \end{bmatrix} \tag{2.14}$$

hold.

Proof. If $A^- \in A\{1,3\}$ and $S^- \in A\{1,3\}$, then $AA^{(1,3)} = AA^\dagger$ and $SS^{(1,3)} = SS^\dagger$. In these cases,

$$r\left(M^* \begin{bmatrix} P_A & 0 \\ 0 & P_S \end{bmatrix} \right) = r\begin{bmatrix} P_AA & P_AB \\ P_SC & P_SD \end{bmatrix}$$
$$= r\begin{bmatrix} A & 0 & A & B \\ 0 & S & C & D \end{bmatrix} - r(A) - r(S) \quad \text{(by (1.7))}$$
$$= r\begin{bmatrix} A & 0 & 0 & B \\ 0 & D & C & 0 \end{bmatrix} - r(A) - r(S)$$
$$= r[\, A,\, B\,] + r[\, C,\, D\,] - r(A) - r(\, D - CA^{(1,3)}B \,).$$

Substituting this into (2.13) results in

$$\min_{M^- \in M\{1,3\}} r[\, M^- - N(A^-) \,] = r[\, A,\, B\,] + r[\, C,\, D\,] - r(A) - r(\, D - CA^-B \,). \tag{2.15}$$

Substituting (1.14) into (2.15) gives

$$\min_{A^- \in A\{1,3\},\, M^- \in M\{1,3\}} r[\, M^- - N(A^-) \,] = \max\{\, r_1, \quad r_2 \,\},$$

where

$$r_1 = r[\, A,\, B\,] + r[\, C,\, D\,] - r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix},\ r_2 = r[\, A,\, B\,] + r[\, C,\, D\,] - r(A) - r\begin{bmatrix} B \\ D \end{bmatrix}.$$

Let the right-hand side be zero, we see that there are $A^- \in A\{1,3\}$ and $S^- \in S\{1,3\}$ such that $N(A^-) \in M\{1,3\}$ if and only if

$$(2.16)\quad r[\,A,\, B\,] + r[\,C,\, D\,] = r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} \text{ and } r[\,A,\, B\,] + r[\,C,\, D\,] \leqslant r(A) + r\begin{bmatrix} B \\ D \end{bmatrix}.$$

Note that

$$r[\,A,\, B\,] + r[\,C,\, D\,] \geqslant r(A) + r[\,C,\, D\,] \geqslant r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix}.$$

Hence, the first equality in (2.16) is equivalent to

$$\mathscr{R}(B) \subseteq \mathscr{R}(A) \text{ and } r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} = r[\,AA^*,\, A^*B\,] + r[\,C,\, D\,],$$

and the second inequality in (2.16) is equivalent to $r[\,C,\, D\,] \leqslant r\begin{bmatrix} B \\ D \end{bmatrix}$. Thus (2.16) is equivalent to (2.14). □

Similarly, the following two theorems can be derived from (1.22) and (1.15).

Theorem 2.8. *Let M and $N(A^-)$ be as given in* (1.1) *and* (1.5), *respectively. Then*

$$(2.17)\quad \min_{M^- \in M\{1,4\}} r[\,M^- - N(A^-)\,] = r\begin{bmatrix} Q_AA^* & Q_AC^* \\ Q_SB^* & Q_SD^* \end{bmatrix}.$$

Hence the following statements are equivalent:

(a) $N(A^-) \in M\{1,4\}$.

(b) *The g-inverses A^- and S^- in $N(A^-)$ satisfy $Q_AA^* = 0$, $Q_AC^* = 0$, $Q_SB^* = 0$ and $Q_SD^* = 0$.*

(c) *$\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*)$, $\mathscr{R}(B^*) \subseteq \mathscr{R}(S^*)$ and the g-inverses A^- and S^- in $N(A^-)$ satisfy $A^- \in A\{1,4\}$ and $S^- \in S\{1,4\}$.*

Theorem 2.9. *Let M and $N(A^-)$ be as given in* (1.1) *and* (1.5), *respectively. Then there are $A^- \in A\{1,4\}$ and $S^- \in S\{1,4\}$ such that $N(A^-) \in M\{1,4\}$ if and only if*

$$\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*),\ \mathscr{R}\begin{bmatrix} AA^* \\ CA^* \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} B \\ D \end{bmatrix} = \{0\} \text{ and } r\begin{bmatrix} B \\ D \end{bmatrix} \leqslant r[\,C,\, D\,]$$

hold.

Note that $X = A^\dagger$ if and only if $X \in A\{1,2\}$, $X \in A\{1,3\}$ and $X \in A\{1,4\}$. Hence the following result is derived from Theorems 2.4, 2.6 and 2.8.

Theorem 2.10 ([1]). *Let M and $N(A^-)$ be as given in* (1.1) *and* (1.5). *Then the following statements are equivalent*:

(a) $N(A^-) = M^\dagger$.

(b) *$\mathscr{R}(B) \subseteq \mathscr{R}(A)$, $\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*)$, $\mathscr{R}(C) \subseteq \mathscr{R}(S)$, $\mathscr{R}(B^*) \subseteq \mathscr{R}(S^*)$ and the g-inverses A^- and S^- in $N(A^-)$ satisfy $A^- = A^\dagger$ and $S^- = S^\dagger$.*

3. The Generalized Schur complement in idempotent matrix. Consider the square block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{3.1}$$

where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{k\times m}$ and $D \in \mathbb{C}^{k\times k}$. Then this matrix is idempotent, namely, $M^2 = M$, if and only if

$$A = A^2 + BC, \quad B = AB + BD, \quad C = CA + DC, \quad D = CB + D^2. \tag{3.2}$$

From (3.2), we find that the square of the generalized Schur complement $S = D - CA^- B$ can be written as

$$\begin{aligned} S^2 &= D^2 - DCA^-B - CA^-BD + CA^-BCA^-B \\ &= D - CB - (C - CA)A^-B - CA^-(B - AB) + CA^-(A - A^2)A^-B \\ &= S - C(I_m - A^-A)(I_m - AA^-)B + C(A^-AA^- - A^-)B. \end{aligned} \tag{3.3}$$

Hence,

$$S^2 = S \Leftrightarrow C(I_m - A^-A)(I_m - AA^-)B = C(A^-AA^- - A^-)B. \tag{3.4}$$

The right-hand side is a quadratic equation with respect to A^-. Hence it is difficult to show the existence of A^- satisfying this equation. Instead, we first consider the existence of A^- satisfying

$$C(I_m - A^-A)(I_m - AA^-)B = 0 \quad \text{and} \quad C(A^-AA^- - A^-)B = 0.$$

Notice that if there is A^- satisfying $C(I_m - A^-A)(I_m - AA^-)B = 0$, then the product $G = A^-AA^-$ satisfies

$$G \in A\{1,2\} \quad \text{and} \quad C(I_m - GA)(I_m - AG)B = C(GAG - G)B = 0.$$

In such a case, $D - CGB$ is idempotent by (3.4). In view of this, we only consider the existence of $A^{(1,2)}$ such that $C(I_m - A^{(1,2)}A)(I_m - AA^{(1,2)})B = 0$.

LEMMA 3.1. *Let M in (3.1) be idempotent. Then*:

(a) *There is $A^{(1,2)}$ such that $C(I_m - A^{(1,2)}A)(I_m - AA^{(1,2)})B = 0$ if and only if*

$$r\begin{bmatrix} A^2 & AB \\ CA & CB \end{bmatrix} \leqslant r\begin{bmatrix} A^2 \\ CA \end{bmatrix} + r[A^2, AB]. \tag{3.5}$$

(b) $C(I_m - A^{(1,2)}A)(I_m - AA^{(1,2)})B = 0$ *holds for any $A^{(1,2)}$ if and only if*

$$\mathscr{R}(B) \subseteq \mathscr{R}(A), \quad \text{or} \quad \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*), \quad \text{or} \quad \begin{bmatrix} A^2 & AB \\ CA & CB \end{bmatrix} = 0. \tag{3.6}$$

Proof. It is well known that the general expression of $A^{(1,2)}$ can be expressed as

$$A^{(1,2)} = (A^\dagger + F_AV)A(A^\dagger + WE_A),$$

where $E_A = I_m - AA^\dagger$ and $F_A = I_m - A^\dagger A$, the two matrices V and W are arbitrary; see [4]. Correspondingly,

$$C(\, I_m - A^{(1,2)} A)(\, I_m - AA^{(1,2)}\,)B = (\, CF_A - CF_AVA\,)(\, E_AB - AWE_AB\,). \tag{3.7}$$

We show that there are V and W such that $(\, CF_A - CF_AVA\,)(\, E_AB - AWE_AB\,) = 0$. It is easy to see

$$r[\,(\, CF_A - CF_AVA\,)(\, P_AB - AWP_AB\,)\,] = r\begin{bmatrix} I_m & E_AB - AWE_AB \\ CF_A - CF_AVA & 0 \end{bmatrix} - m,$$

where

$$\begin{bmatrix} I_m & E_AB - AWE_AB \\ CF_A - CF_AVA & 0 \end{bmatrix} = \begin{bmatrix} I_m & E_AB \\ CF_A & 0 \end{bmatrix} - \begin{bmatrix} A \\ 0 \end{bmatrix} W[\,0,\, E_AB\,] - \begin{bmatrix} 0 \\ CF_A \end{bmatrix} V[\,A,\, 0\,]. \tag{3.8}$$

Applying the following rank formula in [14]

$$\min_{X_1,\, X_2} r(\, A - B_1X_1C_1 - B_2X_2C_2\,) = r\begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} + r[\,A,\, B_1,\, B_2\,] + \max\{\, r_1,\ r_2\,\}, \tag{3.9}$$

where

$$r_1 = r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix},$$

$$r_2 = r\begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}$$

to (3.8) and simplifying yield

$$\begin{aligned} &\min_{V,\, W} r\begin{bmatrix} I_m & E_AB - AWE_AB \\ CF_A - CF_AVA & 0 \end{bmatrix} \\ &= \max\left\{ m,\ \ m + r\begin{bmatrix} A^2 & AB \\ CA & CB \end{bmatrix} - r\begin{bmatrix} A^2 \\ CA \end{bmatrix} - r[\,A^2,\, AB\,] \right\}. \end{aligned}$$

Hence

$$\begin{aligned} &\min_{A^{(1,2)}} r[\, C(I_m - A^{(1,2)}A)(I_m - AA^{(1,2)})B\,] \\ &= \max\left\{ 0,\ \ r\begin{bmatrix} A^2 & AB \\ CA & CB \end{bmatrix} - r\begin{bmatrix} A^2 \\ CA \end{bmatrix} - r[\,A^2,\, AB\,] \right\}. \end{aligned}$$

Letting the right-side be zero yields (a). Applying the following rank formula in [14]

$$\max_{X_1,\, X_2} r(\, A - B_1X_1C_1 - B_2X_2C_2\,)$$
$$= \min\left\{ r\begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix},\ \ r[\, A,\, B_1,\, B_2\,],\ \ r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix},\ \ r\begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} \right\}$$

to (3.8) and simplifying yield

$$\max_{V,\, W} r\begin{bmatrix} I_m & E_AB - AWE_AB \\ CF_A - CF_AVA & 0 \end{bmatrix}$$
$$= m + \min\left\{ r\begin{bmatrix} A \\ C \end{bmatrix} - r(A),\ \ r[\, A,\, B\,] - r(A),\ \ r\begin{bmatrix} A^2 & AB \\ CA & CB \end{bmatrix} \right\}.$$

Hence,

$$\max_{A^{(1,2)}} r[\, C(I_m - A^{(1,2)}A)(I_m - AA^{(1,2)})B\,]$$
$$= \min\left\{ r\begin{bmatrix} A \\ C \end{bmatrix} - r(A),\ \ r[\, A,\, B\,] - r(A),\ \ r\begin{bmatrix} A^2 & AB \\ CA & CB \end{bmatrix} \right\}.$$

Letting the right-side be zero leads to (b). □

Applying Theorem 3.1 to (3.4) gives us the following result.

Theorem 3.2. *Let M in* (3.1) *be idempotent. Then*:

(a) *There is $A^{(1,2)}$ such that the generalized Schur complement $D - CA^{(1,2)}B$ is idempotent if and only if* (3.5) *holds.*

(b) *The generalized Schur complement $D - CA^{(1,2)}B$ is idempotent for any $A^{(1,2)}$ if and only if* (3.6) *holds.*

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