Professor Yanai and Multivariate Analysis

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Professor Yanai in 1992 (Puntanen, Styan, and Isotalo, 2011, p. 307)

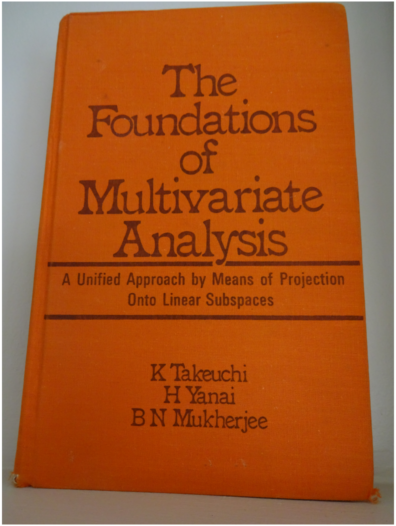
Projectors and SVD

Common threads running through them are:

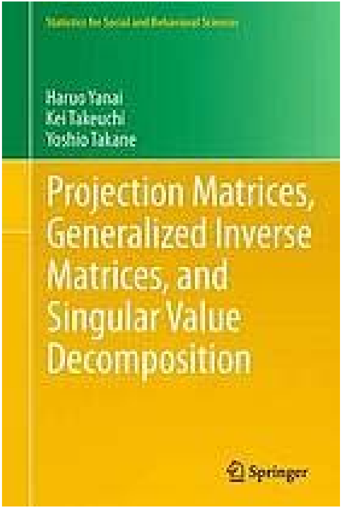
- **projectors**,
- **singular value decomposition (SVD)**,

which are main subject matters of Yanai, Takeuchi, and Takane (2011).

Takeuchi, Yanai, and Mukherjee (1982): The Foundations of Multivariate Analysis



Yanai, Takeuchi, and Takane (2011): Projection matrices, generalized inverse matrices, and singular value decomposition



Topics Covered

- (1) Constrained principal component analysis (CPCA)
- (2) Khatri's lemma
- (3) The Wedderburn-Guttman theorem
- (4) Ridge operators
- (5) Constrained canonical correlation analysis
- (6) Causal inferences

Orthogonal Projectors

- Sp($\mathbf{X}$): The space spanned by column vectors of $\mathbf{X}$.
- Ker($\mathbf{X}'$): The orthogonal complement subspace to Sp($\mathbf{X}$).
- Orthogonal projectors onto Sp($\mathbf{X}$): $\mathbf{P}_X = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$.
- Orthogonal projectors onto Ker($\mathbf{X}'$): $\mathbf{Q}_X = \mathbf{I} - \mathbf{P}_X$.
- Basic properties:

$$\mathbf{P}_X' = \mathbf{P}_X,\ \mathbf{Q}_X' = \mathbf{Q}_X \text{ (symmetric).}$$
$$\mathbf{P}_X^2 = \mathbf{P}_X,\ \mathbf{Q}_X^2 = \mathbf{Q}_X \text{ (idempotent).}$$
$$\mathbf{P}_X\mathbf{Q}_X = \mathbf{Q}_X\mathbf{P}_X = \mathbf{O} \text{ (orthogonal).}$$

K-Orthogonal Projectors

- Let $\mathbf{K}$ be an *nnd* matrix such that $\text{rank}(\mathbf{KX}) = \text{rank}(\mathbf{X})$.
- K-orthogonal projectors: $\mathbf{P}_{X/K} = \mathbf{X}(\mathbf{X}'\mathbf{KX})^{-}\mathbf{X}'\mathbf{K}$, and $\mathbf{Q}_{X/K} = \mathbf{I} - \mathbf{P}_{X/K}$.
- Basic properties:
  $(\mathbf{KP}_{X/K})' = \mathbf{KP}_{X/K}$, $(\mathbf{KQ}_{X/K})' = \mathbf{KQ}_{X/K}$ (K-symmetric).
  $\mathbf{P}^2_{X/K} = \mathbf{P}_{X/K}$, $\mathbf{Q}^2_{X/K} = \mathbf{Q}_{X/K}$ (idempotent).
  $\mathbf{P}'_{X/K}\mathbf{KQ}_{X/K} = \mathbf{Q}'_{X/K}\mathbf{KP}_{X/K} = \mathbf{O}$ (K-orthogonal).

CPCA: Two Phases

- External Analysis and Internal Analysis.
- External Analysis: Decomposes the main data matrix according to the external information abut the row and columns of the data matrix $\Longrightarrow$ projection.
- Internal Analysis: Further analyses of decomposed matrices into components $\Longrightarrow$ SVD (singular value decomposition

External Analysis

- **Y**: The main data matrix.
- **G**: The row (left-hand) side information matrix.
- **H**: The column (right-hand) side information matrix.
- The basic decomposition:

$$\mathbf{Y} = \mathbf{P}_G\mathbf{Y}\mathbf{P}_H + \mathbf{Q}_G\mathbf{Y}\mathbf{P}_H + \mathbf{P}_G\mathbf{Y}\mathbf{Q}_H + \mathbf{Q}_G\mathbf{Y}\mathbf{Q}_H.$$

- A similar decomposition with K-orthogonal projectors.

Finer Decompositions (1)

- $\mathbf{G} = [\mathbf{M}, \mathbf{N}]$.
- (1) $\mathbf{P}_G = \mathbf{P}_M + \mathbf{P}_N \Leftrightarrow \mathbf{M}'\mathbf{N} = \mathbf{O}$.
- (2) $\mathbf{P}_G = \mathbf{P}_M + \mathbf{P}_N - \mathbf{P}_M\mathbf{P}_N \Leftrightarrow \mathbf{P}_M\mathbf{P}_N = \mathbf{P}_N\mathbf{P}_M$.
- (3) $\mathbf{P}_G = \mathbf{P}_M + \mathbf{P}_{Q_M N} = \mathbf{P}_N + \mathbf{P}_{Q_N M}$.
- (4) $\mathbf{P}_G = \mathbf{P}_{M/Q_N} + \mathbf{P}_{N/Q_M} \Leftrightarrow \text{rank}(\mathbf{G}) = \text{rank}(\mathbf{M}) + \text{rank}(\mathbf{N})$.
- (5) $\mathbf{P}_G = \mathbf{P}_{GA} + \mathbf{P}_{G(G'G)^- B} \Leftrightarrow \mathbf{A}'\mathbf{B} = \mathbf{O}$,
  $\text{Sp}(\mathbf{A}) \oplus \text{Sp}(\mathbf{B}) = \text{Sp}(\mathbf{G}')$.
- Analogous decompositions for $\mathbf{P}_H$, $\mathbf{P}_{G/K}$, and $\mathbf{P}_{H/L}$.

Finer Decompositions (2): Explanations

- (1) **M** and **N** are mutually orthogonal.
- (2) **M** and **N** are mutually orthogonal, except their common space. (ANOVA w/o interactions).
- (3) Fit one first and the other to the residuals.
- (4) **M** and **N** are disjoint. Fit both simultaneously.
- (5) A matrix of regression coefficients **C** constrained by $\mathbf{C} = \mathbf{AC}^*$ or by $\mathbf{B}'\mathbf{C} = \mathbf{O}$.

Internal Analysis

- PCA of terms obtained by the external analysis of $\mathbf{Y}$, e.g., $\mathbf{P}_G\mathbf{Y}\mathbf{P}_H$, which amounts to SVD($\mathbf{P}_G\mathbf{Y}\mathbf{P}_H$).

Khatri's Lemma (1)

- Constrained Correspondence Analysis (CCA).
- $\mathbf{U}$: The row representation matrix. (We consider only the row side.)
- Two ways of constraining $\mathbf{U}$: (1) $\mathbf{U} = \mathbf{A}\mathbf{U}^*$, and (2) $\mathbf{B}'\mathbf{U} = \mathbf{O}$.
- $\mathbf{P}_A = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}' = \mathbf{I} - \mathbf{B}(\mathbf{B}'\mathbf{B})^-\mathbf{B}' = \mathbf{Q}_B$.
- What happens if non-identity metric $\mathbf{K}$ is used?
- Let $\mathbf{A}$ $(p \times r)$ and $\mathbf{B}$ $(p \times (p - r))$ be matrices such that rank$(\mathbf{A}) = r$, rank$(\mathbf{B}) = p - r$, and $\mathbf{A}'\mathbf{B} = \mathbf{O}$. Then $\mathbf{I} = \mathbf{A}(\mathbf{A}'\mathbf{K}\mathbf{A})^{-1}\mathbf{A}\mathbf{K} + \mathbf{K}^{-1}\mathbf{B}(\mathbf{B}'\mathbf{K}^{-1}\mathbf{B})^{-1}\mathbf{B}'$ (Khatri, 1966).

Further Remarks

- An alternative expression:
  $\mathbf{K} = \mathbf{KA}(\mathbf{A}'\mathbf{KA})^{-1}\mathbf{AK} + \mathbf{B}(\mathbf{B}'\mathbf{K}^{-1}\mathbf{B})^{-1}\mathbf{B}'$.
- Useful for rewriting Q-type projectors into P-type.

Some Extensions

- Let $\mathbf{A}$ $(p \times r)$ and $\mathbf{B}$ $(p \times (p - r))$ be matrices such that $\text{rank}(\mathbf{A}) = r$ and $\text{rank}(\mathbf{B}) = p - r$, and let $\mathbf{M}$ and $\mathbf{N}$ be *nnd* matrices such that
  (i) $\mathbf{A}'\mathbf{M}\mathbf{N}\mathbf{B} = \mathbf{O}$,
  (ii) $\text{rank}(\mathbf{M}\mathbf{A}) = \text{rank}(\mathbf{A})$,
  (iii) $\text{rank}(\mathbf{N}\mathbf{B}) = \text{rank}(\mathbf{B})$.
  Then,
  $$\mathbf{I} = \mathbf{A}(\mathbf{A}'\mathbf{M}\mathbf{A})^{-}\mathbf{A}'\mathbf{M} + \mathbf{N}\mathbf{B}(\mathbf{B}'\mathbf{N}\mathbf{B})^{-}\mathbf{B}'.$$
- Reduces to the original lemma when $\mathbf{M} = \mathbf{K}$ and $\mathbf{N} = \mathbf{K}^{-1}$.

The WG Theorem

- Let $\mathbf{Y}$ $(n \times p)$ be of rank r, and let $\mathbf{A}$ $(n \times s)$ and $\mathbf{B}$ $(p \times s)$ be such that $\mathbf{A}'\mathbf{Y}\mathbf{B}$ is invertible.
- Then,

$$\begin{aligned}\text{rank}(\mathbf{Y}_1) &= \text{rank}(\mathbf{Y}) - \text{rank}(\mathbf{Y}\mathbf{B}(\mathbf{A}'\mathbf{Y}\mathbf{B})^{-1}\mathbf{A}'\mathbf{Y}) \\ &= \text{rank}(\mathbf{Y}) - \text{rank}(\mathbf{A}'\mathbf{Y}\mathbf{B}) = r - s,\end{aligned}$$

  where

$$\mathbf{Y}_1 = \mathbf{Y} - \mathbf{Y}\mathbf{B}(\mathbf{A}'\mathbf{Y}\mathbf{B})^{-1}\mathbf{A}'\mathbf{Y}.$$

- Wedderburn (1934) for $s = 1$. Guttman (1944) for $s > 1$. Guttman (1957) reverse.

The Generalized WG Theorem

- When $\mathbf{A}'\mathbf{YB}$ is not invertible, can we replace it by a generalized inverse?
- Yes, but it requires a condition.
- A rank additivity (subtractivity) problem?

$$\text{rank}(\mathbf{Y} - \mathbf{YB}(\mathbf{A}'\mathbf{YB})^{-}\mathbf{A}'\mathbf{Y}) = \text{rank}(\mathbf{Y}) - \text{rank}(\mathbf{YB}(\mathbf{A}'\mathbf{YB})^{-}\mathbf{A}'\mathbf{Y}). \quad (1)$$

- Does the following always hold?

$$\text{rank}(\mathbf{YB}(\mathbf{A}'\mathbf{YB})^{-}\mathbf{A}'\mathbf{Y}) = \text{rank}(\mathbf{A}'\mathbf{YB}) \quad (2)$$

- No. Tian and Styan (2009) showed the following always holds:

$$\text{rank}(\mathbf{Y} - \mathbf{YB}(\mathbf{A}'\mathbf{YB})^{-}\mathbf{A}'\mathbf{Y}) = \text{rank}(\mathbf{Y}) - \text{rank}(\mathbf{A}'\mathbf{YB}). \quad (3)$$

- (2) requires a condition, as does (1).

The *ns* Condition

- Let $\mathbf{C} = \mathbf{B}(\mathbf{A}'\mathbf{Y}\mathbf{B})^{-}\mathbf{A}'$.
- The *ns* condition for (1)to hold is:

$$\mathbf{YCYCY} = \mathbf{YCY}.$$

- Equivalent conditions:
  $(\mathbf{YCYY}^{-})^2 = \mathbf{YCYY}^{-} \Leftrightarrow (\mathbf{Y}^{-}\mathbf{YCY})^2 = \mathbf{Y}^{-}\mathbf{YCY}$.
- $(\mathbf{YC})^2 = \mathbf{YC}$ or $(\mathbf{CY})^2 = \mathbf{CY}$ (sufficient but not necessary).
- $\mathbf{CYC} = \mathbf{C}$ (sufficient but not necessary). Even stronger than idempotency of $\mathbf{YC}$ or $\mathbf{CY}$.

The WG Decomposition

- $\mathbf{Y} = \mathbf{YB}(\mathbf{A}'\mathbf{YB})^{-}\mathbf{A}'\mathbf{Y} + (\mathbf{Y} - \mathbf{YB}(\mathbf{A}'\mathbf{YB})^{-}\mathbf{A}'\mathbf{Y})$.
- Let $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ be matrices such that
  (i) $\mathrm{Sp}(\tilde{\mathbf{A}}) \subset \mathrm{Sp}(\mathbf{Y})$,
  (ii) $\mathrm{Sp}(\tilde{\mathbf{B}}) \subset \mathrm{Sp}(\mathbf{Y}')$,
  (iii) $\mathrm{rank}(\mathbf{A}'\mathbf{YB}) + \mathrm{rank}(\tilde{\mathbf{B}}'\mathbf{Y}^{-}\tilde{\mathbf{A}}) = \mathrm{rank}(\mathbf{Y})$,
  (iv) $\mathbf{A}'\mathbf{YY}^{-}\tilde{\mathbf{A}} = \mathbf{A}'\tilde{\mathbf{A}} = \mathbf{O}$,
  (v) $\tilde{\mathbf{B}}'\mathbf{Y}^{-}\mathbf{YB} = \tilde{\mathbf{B}}'\mathbf{B} = \mathbf{O}$.
- Then, $\mathbf{Y} = \mathbf{YB}(\mathbf{A}'\mathbf{YB})^{-}\mathbf{A}'\mathbf{Y} + \tilde{\mathbf{A}}(\tilde{\mathbf{B}}'\mathbf{Y}^{-}\tilde{\mathbf{A}})^{-}\tilde{\mathbf{B}}'$.

Ridge Operator: Definition

- $\mathbf{R}_X(\lambda) = \mathbf{X}(\mathbf{X}'\mathbf{X} + \lambda \mathbf{P}_{X'})^{-}\mathbf{X}'$, where $\mathbf{P}_{X'} = \mathbf{X}'(\mathbf{X}\mathbf{X}')^{-}\mathbf{X}$ is the orthogonal projector onto $\text{Sp}(\mathbf{X}')$. ($\mathbf{P}_{X'} = \mathbf{I}$ if $\mathbf{X}$ is columnwise nonsingular.)
- The ridge LS estimation $\min_{\mathbf{c}} = \phi_\lambda(\mathbf{c})$, where $\phi_\lambda(\mathbf{c}) = \text{SS}(\mathbf{e}) + \lambda \text{SS}(\mathbf{c})_{P_{X'}}$ and $\mathbf{e} = \mathbf{y} - \mathbf{X}\mathbf{c}$. (We assume, w/o loss of generality, that $\text{Sp}(\mathbf{c}) \subset \text{Sp}(\mathbf{X}')$.)

Ridge Operator: Some Properties

- Let $\mathbf{S}_X(\lambda) = \mathbf{I} - \mathbf{R}_X(\lambda)$.
- $\mathbf{R}_X(\lambda)$ and $\mathbf{S}_X(\lambda)$ have properties similar to those of $\mathbf{P}_X$ and $\mathbf{Q}_X$.
- For example:
  $\mathbf{R}_X(\lambda)\mathbf{K}_X(\lambda)\mathbf{R}_X(\lambda) = \mathbf{R}_X(\lambda)$ (i.e., $\mathbf{K}_X(\lambda) = \mathbf{R}_X(\lambda)^+$.),
  $\mathbf{R}_X(\lambda) - \mathbf{R}_X(\lambda)^2 = \mathbf{R}_X(\lambda)\mathbf{S}_X(\lambda) = \mathbf{S}_X(\lambda)\mathbf{R}_X(\lambda) \geq \mathbf{O}$.
  $\mathbf{R}_X(\lambda)\mathbf{K}_X(\lambda) = \mathbf{P}_X$, etc.
- Similar decompositions of $\mathbf{R}_X(\lambda)$ to those of $\mathbf{P}_X$.

Ridge Metric Matrix

- Ridge metric matrix: $\mathbf{K}_X(\lambda) = \mathbf{P}_X + \lambda(\mathbf{XX}')^+$.
- Then, $\mathbf{R}_X(\lambda)$ can be rewritten as:

$$\mathbf{R}_X(\lambda) = \mathbf{X}(\mathbf{X}'\mathbf{K}_X(\lambda)\mathbf{X})^-\mathbf{X}'.$$

Generalized Ridge Operator

- Generalized ridge operator:
  $\mathbf{R}_X^{(W,L)}(\lambda) = \mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X} + \lambda\mathbf{L})^{-}\mathbf{X}'\mathbf{W}$, where $\mathbf{L}$ is an *nnd* matrix such that $\text{Sp}(\mathbf{L}) \subset \text{Sp}(\mathbf{X}')$, and $\mathbf{W}$ is an *nnd* matrix such that $\text{rank}(\mathbf{W}\mathbf{X}) = \text{rank}(\mathbf{X})$.
- Generalized ridge metric matrix:
  $\mathbf{K}_X^{(W,L)}(\lambda) = \mathbf{P}_X + \lambda\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X})^{-}\mathbf{L}(\mathbf{X}'\mathbf{W}\mathbf{X})^{-}\mathbf{X}'\mathbf{W}$.
- Then, $\mathbf{R}_X^{(W,L)}(\lambda) = \mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{K}_X^{(W,L)}(\lambda)\mathbf{X})^{-}\mathbf{X}'\mathbf{W}$.

Decompositions of Total Association

- Total association between $\mathbf{X}$ and $\mathbf{Y}$: $\text{tr}(\mathbf{P}_X\mathbf{P}_Y)$.
- $\mathbf{X} = \mathbf{M} + \mathbf{N}$, $\mathbf{M}'\mathbf{N} = \mathbf{O}$ does not guarantee $\mathbf{P}_X = \mathbf{P}_M + \mathbf{P}_N$.
- cf. $\mathbf{X} = [\mathbf{M}, \mathbf{N}]$, $\mathbf{M}'\mathbf{N} = \mathbf{O}$ leads to $\mathbf{P}_X = \mathbf{P}_M + \mathbf{P}_N$.
- We need orthogonal decompositions of $\mathbf{P}_X$ and $\mathbf{P}_Y$ to derive additive decompositions of the total association.

Two Orthogonal Decompositions of Projectors

- (1) Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{W}$ be matrices such that $\mathrm{Sp}(\mathbf{A}) = \mathrm{Ker}(\mathbf{H}'\mathbf{X}'\mathbf{P}_G\mathbf{X})$, $\mathrm{Sp}(\mathbf{B}) = \mathrm{Ker}(\mathbf{H}'\mathbf{X}'\mathbf{Q}_G\mathbf{X})$, and $\mathrm{Sp}(\mathbf{W}) = \mathrm{Ker}(\mathbf{X}'\mathbf{G})$. Then,

$$\mathbf{P}_{[X,G]} = \mathbf{P}_{P_G XH} + \mathbf{P}_{P_G XA} + \mathbf{P}_{Q_G XH} + \mathbf{P}_{Q_G XB} + \mathbf{P}_{GW}.$$

- (2) Let $\mathbf{K}$, $\mathbf{U}$, and $\mathbf{V}$ be matrices such that $\mathrm{Sp}(\mathbf{K}) = \mathrm{Ker}(\mathbf{H}'\mathbf{X}'\mathbf{X})$, $\mathrm{Sp}(\mathbf{U}) = \mathrm{Ker}(\mathbf{G}'\mathbf{X}\mathbf{H})$, and $\mathrm{Sp}(\mathbf{V}) = \mathrm{Ker}(\mathbf{G}'\mathbf{X}\mathbf{K})$. Then,

$$\mathbf{P}_{[X,G]} = \mathbf{P}_{P_{XH} G} + \mathbf{P}_{XHU} + \mathbf{P}_{P_{XK} G} + \mathbf{P}_{XKV} + \mathbf{P}_{Q_X G}.$$

Constrained Canonical Correlation Analysis

- Similar decompositions of $\mathbf{P}_{[\mathbf{Y},\mathbf{G}_Y]}$.
- Take one term each from a decomposition of $\mathbf{P}_{[\mathbf{X},\mathbf{G}_X]}$ and that of $\mathbf{P}_{[\mathbf{Y},\mathbf{G}_Y]}$, apply SVD to the product of the two, e.g.,

$$\mathrm{SVD}(\mathbf{P}_{\mathbf{Q}_{G_X} XH_X}\mathbf{P}_{YH_Y U_Y}).$$

Confounding Variables

- Causal inferences without randomization. How to eliminate the effects of confounding variables.
- $\mathbf{y}$: The dependent variable.
- $\mathbf{x}$: The independent variable.
- $\mathbf{U}$: The confounding variables.
- Regression analysis (1): $\mathbf{y} = \mathbf{x}a_1 + \mathbf{Uc} + \mathbf{e}_1$. The OLS estimate of $\mathbf{x}a_1$ is given by

$$\mathbf{x}\hat{a}_1 = \mathbf{P}_{x/Q_u}\mathbf{y} \tag{4}$$

.

- On the other hand, consider the regression of $\mathbf{x}$ onto $\mathbf{U}$, i.e., $\mathbf{x} = \mathbf{Ud} + \mathbf{e}_2$. The OLS estimate of $\mathbf{Ud}$ is given by

$$\mathbf{U}\hat{\mathbf{d}} = \mathbf{P}_U\mathbf{x}. \tag{5}$$

Linear Propensity Scores

- We call $\mathbf{P}_U\mathbf{x}$ linear propensity scores. Residuals from the above regression $\mathbf{Q}_U\mathbf{x}$ represent the portions of $\mathbf{x}$ left unaccounted for by $\mathbf{U}$.
- We next consider using $\mathbf{P}_U\mathbf{x}$ instead of $\mathbf{U}$ in the first regression, i.e., $\mathbf{y} = \mathbf{x}a_2 + \mathbf{P}_U\mathbf{x}b + \mathbf{e}_3$. the OLS estimate of $\mathbf{x}a_2$ is given by

$$\mathbf{x}\hat{a}_2 = \mathbf{P}_{x/Q_{P_U x}}\mathbf{y}, \tag{6}$$

  where $\mathbf{Q}_{P_U x} = \mathbf{I} - \mathbf{P}_U\mathbf{x}(\mathbf{x}'\mathbf{P}_U\mathbf{x})^{-1}\mathbf{x}'\mathbf{P}_U$.
- Since $\mathbf{Q}_{P_U x}\mathbf{x} = \mathbf{x} - \mathbf{P}_U\mathbf{x}(\mathbf{x}'\mathbf{P}_U\mathbf{x})^{-1}\mathbf{x}'\mathbf{P}_U\mathbf{x} = \mathbf{Q}_U\mathbf{x}$, we obtain

$$\mathbf{P}_{x/Q_{P_U x}}\mathbf{y} = \mathbf{P}_{x/Q_U}\mathbf{y}. \tag{7}$$

  This means (4) and (6) are equivalent.

Instrumental Variable (IV) Estimation

- Regression analysis: $\mathbf{y} = \mathbf{x}a_3 + \mathbf{e}_4$. The IV estimate of $\mathbf{x}a_3$ with $\mathbf{z} = \mathbf{Q}_U x$ as the IV is given by

$$\mathbf{x}\hat{a}_3 = \mathbf{P}_{x/P_z}\mathbf{y} = \mathbf{P}_{x/Q_U}\mathbf{y}. \tag{8}$$

- Since $\mathbf{P}_z = \mathbf{Q}_U\mathbf{x}(\mathbf{x}'\mathbf{Q}_U\mathbf{x})^{-1}\mathbf{x}'\mathbf{Q}_U$ and $\mathbf{x}'\mathbf{P}_z = \mathbf{x}'\mathbf{Q}_U$, this is identical to (4) and (6).

- It can also be easily verified that $\mathbf{z}$ defined above satisfies the following properties required of a IV:
  (i) $\mathbf{z}'\mathbf{U} = \mathbf{0}$ ($\mathbf{z}$ and $\mathbf{U}$ are uncorrelated),
  (ii) $\mathbf{z}'\mathbf{x} \neq 0$ ($\mathbf{z}$ and $\mathbf{x}$ are correlated),
  (iii) $\mathbf{z}'\mathbf{Q}_{[U,x]}\mathbf{y} = 0$ (*i.e.*, $\mathbf{z}$ has a predictive power on $\mathbf{y}$ only through $\mathbf{x}$).
- (i) and (ii) are trivial. That it also satisfies (3) can be seen from:

$$\mathbf{z}'\mathbf{Q}_{[U,x]}\mathbf{y} = \mathbf{x}'\mathbf{Q}_U\mathbf{Q}_{[U,x]}\mathbf{y} = \mathbf{x}'\mathbf{Q}_{[U,x]}\mathbf{y} = 0. \tag{9}$$

Thanks

Thanks for your attention.