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Abstract: Square tables in which some of the elements located symmetrically about the diagonal are not equal to each other are called asymmetric tables. Asymmetric tables arise in a variety of different ways. They can be roughly classified into two groups, one arising from measurements of antisymmetric relationships, and the other of proximity relationships. A variety of models have been proposed to capture the asymmetry in such tables. In this article we briefly discuss some of the representative models of asymmetry for each of the two types of data.

Scaling asymmetric tables

Tables with the same number of rows and columns are called square tables. In square tables corresponding rows and columns often represent the same entities (objects, stimuli, variables, etc.). For example, the i^{th} row of the table represents stimulus i, and the i^{th} column also represents the same stimulus i. Let x_{ij} denote the element in the i^{th} row and the j^{th} column of the table. (We often call it ij^{th} element of the table.) We use X (in matrix form) to denote the entire table collectively. Element x_{ij} indicates (the strength of) some kind of relationship between the row entity (stimulus i) and the column entity (stimulus j). Square tables, in which $x_{ij} \neq x_{ji}$ for some combinations of i and j, are called asymmetric tables. In matrix notation, this is written as $X' \neq X$, where X' indicates the transpose of X.

Asymmetric tables arise in a number of different guises. In some cases the kind of relationship represented in the table is antisymmetric. For example, suppose you have a set of stimuli, and you ask a group of subjects whether they prefer stimulus i or j for each pair of stimuli. Since j cannot be preferred to i if i is preferred to j, the preference choice constitutes an antisymmetric relationship. Let x_{ij} denote the number of times i is preferred to j. Tables representing antisymmetric relationships are usually asymmetric. This type of tables are often skew-symmetric or can easily be turned into one by a simple transformation (e.g., $y_{ij} = \log(x_{ij}/x_{ji})$). In the skew symmetric table, $y_{ji} = -y_{ij}$ ($Y' = -Y$). Skew symmetric data such as the one just described are often represented by the difference between the preference values of the two stimuli involved. Let u_i represent the preference value of stimulus i. Then, $y_{ij} = u_i - u_j$. Case V of Thurstone's law of comparative judgment [8], and Bradley-Terry-Luce (BTL) model [1, 7] are examples of this class of models. The scaling problem here is to find estimates of u_i's given a set of observed values of y_{ij}'s.

Here is an example. The top panel of Table 1 gives observed choice probabilities among four music composers labelled as B, H, M and S. Numbers in the table indicate the proportions of times row composers are preferred to column composers. Let us apply the BTL model to this data set. The second panel of the table gives the skew symmetric table obtained by applying the transformation, $y_{ij} = \log(x_{ij}/x_{ji})$, to the observed choice probabilities. Least squares estimates of preference values for the four composers are obtained by row means of this skew symmetric table. B is the most preferred, M the second, then S, and H the least. Something similar can also be done with Thurstone's Case V model. The only difference it makes is that normal quantile (deviation) scores are obtained, when the matrix of the observed choice probabilities is converted into a skew symmetric matrix. The rest of the procedure remains essentially the same as in the BTL model.

***** Insert table 1 about here *****

Asymmetric tables can also arise from proximity relationships, which are often symmetric. In some cases they exhibit asymmetry, however. For example, you may ask a group of subjects to identify the stimulus presented out of n possible stimuli, and count the number of times stimulus i is "confused" with stimulus j. This is called stimulus recognition (or identification) data, and it is usually asymmetric. There are a number of other examples of asymmetric proximity data such as mobility tables, journal citation data, brand loyalty data, discrete panel data on two occasions, etc. In this case a challenge is how to explain the asymmetry in the tables.

A variety of models have been proposed for asymmetric proximity data. Perhaps the simplest model is the quasi-symmetry model. The quasi-symmetry is characterized by $x_{ij} = a_i b_j c_{ij}$, where a_i and b_j are row and column marginal effects, and $c_{ij} = c_{ji}$

indicates a symmetric similarity between i and j. This model postulates that after removing the marginal effects, the remaining relation is symmetric. (In the special case in which $a_i = b_i$ for all i leads to a full symmetric model.) The quasi-symmetry also satisfies the cycle condition stated as $x_{ij}x_{jk}x_{ki} = x_{ji}x_{kj}x_{ik}$. In some cases, the symmetric similarity parameter, c_{ij}, may further be represented by a simpler model, $c_{ij} = \exp(-d_{ij})$, or $c_{ij} = \exp(-d_{ij}^2)$, where d_{ij} is the Euclidean distance between stimuli i and j represented as points in a multidimensional space.

DEDICOM (DEcomposing DIrectional COMponents, [4]) attempts to explain asymmetric relationships between n stimuli by a smaller number of asymmetric relationships. The DEDICOM model is written as $X = ARA'$, where R is a square asymmetric matrix of order r (capturing asymmetric relationships between r components, where r is assumed much smaller than n), and A is an n by r matrix that relates the latent asymmetric relationships among the r components to the observed asymmetric relationships among the n stimuli. Several algorithms have been developed to fit the DEDICOM model. To illustrate, the DEDICOM model is applied to a table of car switching frequencies among 16 types of cars [4]. (This table indicates frequencies with which a purchase of one type of cars is followed by a purchase of another type by the same consumer.) Table 2 reports the analysis results [5]. Labels of the 16 car types consist of two components. The first three characters mainly indicate size (SUB = subcompact, SMA = small specialty, COM = compact, MID = mid-size, STD = standard, and LUX = luxury), and the fourth character indicates mainly origin or price (D = domestic, C = captive imports, I = imports, L = low price, M = medium price, and S = specialty). The top portion of the table gives the estimated A matrix (normalized so that $A'A = I$), from which we may deduce that the first component (dimension) represents plain large and mid-size cars, the second component represents fancy large cars, and the third represents small/specialty cars. The bottom portion of the table represents the estimated R matrix that captures asymmetry relationships among the three components. There are more switches from 1 to 3, 1 to 2, and 2 to 3 than the other4 way round. This three-component DEDICOM model captures 86.4% of the total SS (sum of squares) in the original data.

***** Insert Table 2 about here *****

Any asymmetric table can be decomposed into the sum of a symmetric matrix (X_s) and a skew symmetric matrix (X_{sk}). That is, $X = X_s + X_{sk}$, where $X_s = (X + X')/2$, and $X_{sk} = (X - X')/2$. The two parts are often analyzed separately. X_s is often analyzed by a symmetric model (such as the inner product model or a distance model like those for c_{ij} described above). X_{sk}, on the other hand, is either treated like a skew symmetric matrix arising from an antisymmetric relationship, or by CASK (Canonical Analysis of SKew symmetric data, [3]). The latter decomposes X_{sk} in the form of AKA' where K consists of 2 by 2 blocks of the form $\begin{pmatrix} 0 & k_l \\ -k_l & 0 \end{pmatrix}$ for the l^{th} block. This representation can be analytically derived from the singular value decomposition of X_{sk}.

Generalized GIPSCAL [6] and HCM (Hermitian Canonical Model, [2]) analyze both parts (X_s and X_{sk}) simultaneously. The former represents X by $B(I_r + K)B'$

(where the BB' part represents X_s and the BKB' part represents X_{sk}) under the assumption that the skew symmetric part of R (i.e., $(R - R')/2$) in DEDICOM is positive definite. The HCM first forms an hermitian matrix, H, by $H = X_s + iX_{sk}$ (where i is a symbol for an imaginary number, $i = \sqrt{-1}$), and obtains the eigenvalue-vector decomposition of H.

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	Observed Choice Probabilities			
	B	H	M	S
B	.500	.895	.726	.895
H	.105	.500	.147	.453
M	.274	.853	.500	.811
S	.105	.547	.189	.500
	Matrix of $y_{ij} = \log(x_{ij}/x_{ji})$			
B	0	2.143	.974	2.143
H	-2.143	0	-1.758	-.189
M	-.974	1.758	0	1.457
S	-2.143	.189	-1.457	0
	Estimated Preference Values			
	1.315	-1.022	.560	-0.853

Table 1: The BTL model applied to preference choice data involving four music composers.

Matrix A			
	Dimension		
Car Class	1	2	3
SUBD	.13	-.02	.36
SUBC	.02	.00	.03
SUBI	.03	.01	.30
SMAD	.01	.03	.53
SMAC	.00	.00	.00
SMAI	.00	.01	.09
COML	.24	-.11	.17
COMM	.10	-.01	.06
COMI	.02	.00	.03
MIDD	.54	.00	.12
MIDI	.02	.00	.02
MIDS	.09	.24	.58
STDL	.68	-.08	-.18
STDM	.32	.67	-.27
LUXD	-.23	.69	.05
LUXI	.00	.02	.01
Matrix R (divided by 1000)			
dim. 1	127	57	78
dim. 2	26	92	23
dim. 3	17	12	75

Table 2: DEDICOM applied to car switching data.