Statistics: Multidimensional Scaling

(1) Zhidong Zhang
zhidong.zhang@mail.mcgill.ca

Department of Psychology
McGill University
Montreal, Canada

Full contact address:

Zhidong Zhang, Ph. D.
Department of Psychology
McGill University
W7/16, Stewart Biological Sciences Bldg.
1205 Dr. Penfield Avenue
Montreal, Quebec
Canada H3A 1B1
Tel. 514-398-4319

(2) Yoshio Takane
takane@takane2.psych.mcgill.ca

Department of Psychology
McGill University
Montreal, Canada

Full contact address

Professor Yoshio Takane
Department of Psychology
McGill University
N7/3, Stewart Biological Sciences Bldg.
1205 Dr. Penfield Avenue
Montreal, Quebec
Canada H3A 1B1
Tel. 514-398-6125

Keywords: multidimensional scaling, Euclidean distance, weighted Euclidean distance, similarity, dissimilarity, individual differences, ideal points, preference, unfolding analysis.

Glossary

Multidimensional scaling (MDS): A set of data analysis techniques for analysis of similarity or dissimilarity data. It is used to represent (dis)similarity data between objects by a variety of distance models.

Distance Models: The models used to represent (dis)similarity data in MDS. Although there are other distance models, the Euclidean distance model is the most popular one used in MDS. The exact definition is described in section 2 of this article.

(Dis)similarity: The term similarity is used to indicate the degree of "likeness" between two objects, while dissimilarity indicates the degree of "unlikeness". For example, red and pink are more similar (less dissimilar) to each other than red and green. (Red and green are more dissimilar (less similar) than red and pink.) In the similarity data a larger value indicates more similar objects, while in the dissimilarity data a larger value indicates more dissimilar objects.

Unfolding analysis: One way of representing individual differences in preference judgments. In unfolding analysis, subjects' ideal objects and actual objects are represented as points in a joint multidimensional space in such a way that the distances between them are as inversely related to the observed preferences as possible.

Abstract

Multidimensional scaling (MDS) is a set of data analysis techniques used to explore the structure of (dis)similarity data. MDS represents a set of objects as points in a multidimensional space in such a way that the points corresponding to similar objects are located close together, while those corresponding to dissimilar objects are located far apart. The investigator then attempts to "make sense" of the derived object configuration by identifying meaningful regions and/or directions in the space. In this article, we first introduce the basic concepts and models of MDS. We then discuss a variety of (dis)similarity data and their scale levels, and the kinds of MDS techniques to be used in specific situations such as individual differences MDS and unfolding analysis.

BODY OF THE ARTICLE

The notion of similarity plays a fundamental role in human cognition (Takane, Jung, and Oshima-Takane, 2009). It serves as an organizing principle by which people categorize, generalize, and classify objects. Multidimensional scaling (MDS) is a set of data analysis techniques for representing (dis)similarity data (similarity or dissimilarity data) by spatial distance models (Takane, 2007). In this article, we explicate the purposes, the mechanism, and the variety of uses of MDS.

This article consists of the following sections: (1) Introduction, via an example, to illustrate the basic roles and the uses of MDS, (2) Distance models, fitting criteria, and the data collection methods, (3) Scale levels and data transformations, (4) Dimensionality selection, (5) Individual differences MDS, (6) Unfolding analysis, and (7) Software for MDS.

1 Introduction

Some objects are more similar (or dissimilar) to each other than others. For example, red and pink are more similar than red and green. MDS represents the similarity or dissimilarity data among the objects by mapping the points (representing the objects) into a multidimensional space in such a way that the distances between them best accord with the observed (dis)similarity data between the objects. In the above example, the points representing red and pink are located closer in the space than the points representing red and green. By virtue of MDS, we can visually inspect the (dis)similarity data among the objects and investigate the principle underlying the organization of the (dis)similarity data.

To further illustrate the role of MDS, let us take a look at the data in Table 1. This table shows dissimilarity data among eight different sports. The names of two sports were presented each time to the subjects, who were asked to indicate the degree of dissimilarity between them on a 11-point rating scale. Entries in the table indicate average dissimilarities between the sports across ten subjects. The eight sports are: 1. baseball, 2. basketball, 3. rugby, 4. soccer, 5. softball, 6. table tennis, 7. tennis, and 8. volleyball. MDS was applied to the table, and the derived object configuration is presented in Figure 1.

***** Insert Table 1 and Figure 1 about here *****

By inspection, it can be readily seen that MDS indeed located points corresponding to similar objects close together, while those corresponding to dissimilar objects far apart. Figure 1 shows that the eight sports are roughly classified into four groups, one consisting of rugby and soccer, the second consisting of volleyball and basketball, the third consisting of baseball and softball, and the fourth consisting of tennis and table tennis. This is consistent with our intuition that the sports within the groups have much in common. The four groups of sports may further be combined in various ways to form larger clusters. For example, baseball, softball, rugby, and soccer might be grouped into one, and the remaining sports (basketball, volleyball, tennis, and table tennis) into the other. Since the first group occupies upper right portions of the configuration, and the second group lower left portions, we may interpret the direction from upper right to lower left contrasting sports that use a big outdoor field with those that require only a medium to small size court. We may also group baseball, softball, tennis, and table tennis into one group, and the remaining ones (rugby, soccer, volleyball and basketball) into the other. The former use a relatively small ball, while the latter a big ball. We may call the direction from upper left to lower right the ball size dimension. MDS, simply stated, is a sort of "gadget" that draws a map like the one presented in Figure 1 based on a set of distance-like quantities (similarity or dissimilarity data) like the ones given in Table 1. The map facilitates our intuitive understanding of the relationships among the objects represented in the map.

2 Distance models, fitting criteria, and the data collection methods

As noted above, MDS represents inter-object (dis)similaities by inter-point distances. While there are a variety of distance models that may be used in MDS, the one most frequently used is the Euclidean distance model. Let x_{ir} denote the coordinate of point i (object i) on dimension r. Then the Euclidean distance between points i and j is calculated by

$$d_{ij} = \{\sum_{r=1}^{R}(x_{ir} - x_{jr})^2\}^{1/2}, \tag{1}$$

where R indicates the dimensionality of the space. Once x_{ir}'s are given, we can locate the points in the space using a Cartesian coordinate system, and we can calculate the distance between them using the above formula. Suppose that $R = 2$, and the coordinates of point 1 on the two dimensions are 3 and 2 ($x_{11} = 3$, and $x_{12} = 2$), and the coordinates of point 2 are 1 and 4 ($x_{21} = 1$, and $x_{22} = 4$). Then these two points can be located as indicated in Figure 2. The Euclidean distance between them can be calculated by $d_{12} = \{(3-1)^2 + (2-4)^2\}^{1/2} = \sqrt{8} \approx 2.828$. MDS locates the points (i.e., finds their coordinates) representing the objects in such a way that the set of distances calculated from the coordinates "best" agree with the observed (dis)similarities between the objects.

***** Insert Figure 2 about here *****

One important feature of the Euclidean distance is that it is invariant over the choice of origin and orientation of coordinate axes. In MDS, we typically place the origin at the centroid of the object configuration, and rotate the configuration in such a way that the coordinate axes represent substantively meaningful attributes. (Note, however, that some distance models used in MDS, e.g., the weighted Euclidean model, do not allow rotation of axes without changing the inter-point distances. See section 5.)

Observed data typically contain a sizable amount of measurement errors, and an exact representation of the data is usually impossible. Rather, we look for the "best" approximation of the observed (dis)similarity data. To make this notion more rigorous, we need to introduce an index that measure the goodness (or badness) of agreement between the observed data and the distance model. This index also serves as a criterion to be optimized in MDS. That is, an MDS procedure systematically looks for the object configuration that maximizes the goodness (or minimizes the badness) of fit of the distance model to the observed (dis)similarity data.

Two broad classes of goodness of fit criteria have been used in MDS. One is the least squares (LS) criterion (Kruskal, 1964a, b), and the other is the maximum likelihood (ML) criterion (Ramsay, 1977, 1982). Although the latter has some appeal for its statistical inference capabilities, the former has been far more predominantly used in MDS for its simplicity and flexibility. Let o_{ij} denote the observed dissimilarity between objects i and j (temporarily assumed to have been measured on a ratio scale; see the next section for scale levels of measurement), and let d_{ij} denote the corresponding distance between points i and j in the Euclidean space. Then, the LS criterion is defined by:

$$\phi(\{x_{ir}\}) = \sum_{i<j}(o_{ij} - d_{ij})^2, \tag{2}$$

where $\{x_{ir}\}$ is a collection of object coordinates. (The LS criterion of the above form is often called "Raw Stress" in the MDS literature.) This is a badness of fit criterion, meaning that a larger value indicates a larger discrepancy between the distance model and the observed dissimilarity data. The LS MDS attempts to find the set of object coordinates $\{x_{ir}\}$ so as to minimize the discrepancy between the observed o_{ij} and the predicted d_{ij}.

A minimization of the LS criterion generally involves a very complicated process because the distance model is not a simple linear function of its parameters (object coordinates). The equations to be satisfied at the minimum of ϕ usually cannot be solved in closed form, and some kind of iterative methods have to be used to solve the equations. In the iterative methods, successive approximations to the final solution are obtained by gradually improving the goodness of fit of the solution, starting from an initial guess, until a sufficiently close approximation is found. See Borg and Groenen (2005) for more details of the algorithms used in MDS.

One potential danger of this kind of iterative optimization procedures is known as the problem of convergence to non-global minima. The LS criterion used in MDS may have multiple local minima, and iterative optimization procedures may be caught up by one of them that are not the true minimum of the criteria we wish to find. Fortunately, the computer nowadays are so powerful that it is not at all unrealistic to obtain multiple solutions starting from many different initial estimates. Multiple solutions may be compared in terms of their goodness of fit, and the best solution can be chosen, which is more likely to be the globally optimal solution we want.

A variety of (dis)similarity measures may be used as input data to MDS, including (dis)similarity ratings, sorting data, confusion data, frequency of co-occurrences, response latency (reaction time data), frequency of social interactions, profile similarity, etc. In the (dis)similarity rating methods, objects are presented in pair to the subjects, who are asked to rate the degree of (dis)similarity between them on a rating scale. (The example presented earlier on dissimilarity among eight sports were collected by this method.) In the sorting method, subjects are given a set of objects and are asked to group them into several groups in terms of their similarity. The number of times two objects are put into same groups is counted over a group of subjects and used as a similarity measure between the objects. See Takane et al. (2009) for more systematic descriptions of the data collection methods used in MDS.

3 Scale levels and data transformations

Scale levels refer to approximate relationships that may be assumed to hold between observed dis(similarity) data and distances. The distinction between different scale levels is important because certain MDS procedures only apply to dis(similarity) data measured on certain scale levels.

There are four scale levels considered in MDS: ratio, interval, log-interval, and ordinal. Let o_{ij} denote the observed dissimilarity between objects i and j, and let d_{ij} denote the corresponding distance. In the ratio scale level, it is assumed that $o_{ij} \approx d_{ij}$, where "$\approx$" means "approximately equal." In this case, d_{ij} can be directly fitted to o_{ij} so as to minimize criterion (2). No data transformation is necessary. However, it is rare to find the ratio-scaled measurement in social science research.

In the interval-scaled measurement, it is assumed that $o_{ij} \approx ad_{ij} + b$, where a is +1 if the data are dissimilarity data, or -1 if they are similarity data (o_{ij} can be either similarity or dissimilarity data), and b is an additive constant. In the case of interval-scaled data, the fitting criterion is generalized into

$$\phi(\{x_{ir}\}, b) = \sum_{i<j} (\pm o_{ij} - d_{ij} - b)^2, \tag{3}$$

and both the object configuration and an optimal value of b have to be estimated that jointly minimize the criterion.

In a log-interval scale, it is assumed that $o_{ij} \approx bd_{ij}^a$, that is, the observed (dis)similarity data are related to underlying distances by a power transformation. When the log is taken on both sides of the above relationship, we obtain $\ln o_{ij} \approx a \ln d_{ij} + \ln b$, which is a linear relationship between log o_{ij} and log d_{ij} similar to the interval scale level, and hence the name log-interval scale. Not many MDS procedures recognize this scale level as such, and the (dis)similarity data at this scale level are often analyzed as mere ordinal scaled data.

In the ordinal scale level, o_{ij} and d_{ij} are assumed to be only monotonically related. That is, $o_{ij} > o_{i'j'}$ implies $d_{ij} \geq d_{i'j'}$ if the data are dissimilarity data, whereas $o_{ij} > o_{i'j'}$ implies $d_{ij} \leq d_{i'j'}$ if the data are similarity data. MDS procedures that are capable of handling ordinal (dis)similarity data are called nonmetric MDS (Shepard, 1962; Kruskal, 1964a, b) and enjoy widest applications. The fitting criterion in this case is modified into:

$$\phi(\{x_{ir}\}, m) = \sum_{i<j} (m(o_{ij}) - d_{ij})^2, \tag{4}$$

where m denotes a monotonic (or an inversely monotonic) transformation. MDS procedures in this case have to find the best monotonic transformation of the ordinal data as well as the object configuration $\{x_{ir}\}$ that jointly minimize the above criterion.

We usually do not know *a priori* exact scale levels that the observed (dis)similarity data satisfy. As a practical strategy, we may start with a weaker assumption, but as soon as we find, as a result of the analysis, that a stronger measurement assumption can be justified, we switch to the stronger assumption. In this way we can get more reliable results while avoiding unaffordable scale level assumptions.

4 Dimensionality Selection

One important decision that has to be made in MDS concerns the dimensionality of the solution space. The dimensionality refers to the number of coordinates needed to locate a point in the spatial representation of objects. There are several considerations that should be taken into account in determining the number of dimensions. The derived object configuration has to fit to the data at hand reasonably well, but should not fit too well. A better fit to the data at hand can generally be achieved by merely increasing the dimensionality of the solution space, and too good a fit may compromise the predictability of the model for future observations. One practical strategy for determining the adequate number of dimensions is to analyze the data under varied dimensionalities, say from 1 to 4, plot the fit value against the dimensionality (this is called a scree plot), and identify the point where the improvement in fit flattens out. Such a point is called an elbow in the scree plot.

Another important consideration in determining the dimensionality of the solution space is the interpretability of derived dimensions. Uninterpretable dimensions are useless and should not be retained (even if they are necessary to account for the observed (dis)similarity data sufficiently well).

In the example of sports data, the Raw stress values were .186 for the unidimensional solution, .020 for the two-dimensional solution, .003 for the three dimensional solution, and .001 for the four dimensional solution. The two-dimensional solution was easily interpretable, whereas the third dimension was not. Thus, the two-dimensional solution was selected as the best solution.

5 Individual differences MDS

So far we have assumed that there is only one set of (dis)similarity data. In many applications of MDS, however, (dis)similarity data are collected from a group of subjects. If no systematic individual differences exist, a single common Euclidean distance model may be fitted to all of them simultaneously, or a single Euclidean distance model is fitted to average (dis)similarity data, as has been done in the example presented earlier. In many situations, however, the assumption of no systematic individual differences is unrealistic. In such a case, each (dis)similarity matrix may be analyzed separately, yielding as many object configurations as there are (dis)similarity matrices. A natural question is how they are related. In most cases, there are both common and unique aspects in (dis)similarity judgments obtained from different individuals. If so, we need a methodology that captures both aspects.

The individual differences (ID) MDS model captures both commonality and individual differences in (dis)similarity judgments (Carroll and Chang, 1970). More specifically, it postulates a common object configuration that applies to all individuals, but that dimensions in the common configuration are differentially weighted by different individuals to give rise to differences in (dis)similarity judgments. The idea of differential weighting of dimensions can be captured by the weighted Euclidean distance model:

$$d_{ijk} = \left\{ \sum_{r=1}^{R} w_{kr}(x_{ir} - x_{jr})^2 \right\}^{1/2}, \tag{5}$$

where d_{ijk} is the distance between points (objects) i and j for individual k, x_{ir} is, as before, the coordinate of object i on dimension r in the common object configuration, and w_{kr} is the weight attached to dimension r by subject k. To eliminate the size indeterminacy between the object configuration and the individual difference weights, the former is typically constrained to satisfy $\sum_{i=1}^{n} x_{ir}^2/n = 1$ for $r = 1, \cdots, R$. In contrast to the simple Euclidean distance model (1), the orientation of the coordinate axes is uniquely determined in the weighted Euclidean distance model. ID MDS estimates both the object coordinates $\{x_{ir}\}$ and the individual differences weights $\{w_{kr}\}$ in such a way that d_{ijk} calculated from them best agree with the observed dissimilarity between objects i and j by subject k.

As an example of ID MDS, let us look at the data in Table 2 obtained from ten subjects. This is the original data from which the data in Table 1 were calculated by averaging them over the ten subjects. Individual differences MDS was applied to the ten dissimilarity matrices. Figure 3 displays the derived two-dimensional common object configuration. In the figure, the sports that use a big ball are located toward right, and those that use a small ball toward left, so that the horizontal axis can be interpreted as representing the ball size dimension. The vertical axis, on the other hand, places the sports that use a big field at the top, and those that use a small court at the bottom, thus contrasting between the two. (As has been alluded to above, coordinates axes are unrotatable in the weighted Euclidean model, so that we don't have to search for meaningful directions in the space. We simply try to interpret the directions of the coordinate axes.) Figure 4 depicts the individual differences weights attached to the two dimensions by different subjects. In the figure two extreme subjects are identified by subject numbers. Subject 6 puts the most emphasis on dimension 1 among all the subjects, whereas subject 10 does so on dimension 2, although in this particular example, the weights are relatively homogeneous, indicating that there are not much differences in the way the two dimensions are evaluated by the different subjects. In some cases, the weights may show interesting patterns of differences that may be related to subjects' background information such as gender, age, level of education, etc., but unfortunately such information is unavailable in the present case. See Takane (2007) for more examples of interesting applications of the ID MDS.

***** Insert Table 2, and Figures 3 and 4 about here *****

6 Unfolding analysis

Individual differences are far more prevalent in preference judgments. Preference data are often analyzed by a variant of MDS called unfolding analysis (Coombs, 1964). In unfolding analysis, each subject is assumed to have an ideal object represented as the subject's ideal point in the same space as actual objects are represented. The distances between the ideal point and the object points are assumed to be inversely related to the subject's preferences on the objects. Let x_{ir} denote the coordinate of object i on dimension r, and y_{jr} the coordinate of subject j's ideal point on dimension r. The Euclidean distance between object point i and ideal point j is calculated by

$$d_{ij} = \left\{ \sum_{r=1}^{R} (x_{ir} - y_{jr})^2 \right\}^{1/2}. \tag{6}$$

The coordinates of the ideal and object points are determined in such a way that the preference values of the objects for a particular subject are a decreasing function of the distances between them. This implies that the closer an object point is to his ideal, the more preferred the object is by that subject. The preference relations are thus regarded as representing similarity relations between the subjects' ideal objects and actual objects. In unfolding analysis, we are given an N by n data matrix obtained from N subjects making preference judgments on n objects. By subjecting the data matrix to unfolding analysis, we obtain two coordinate matrices, one for object points, and the other for subjects' ideal points.

As an example of unfolding analysis, let us look at Table 3. Thirty one subjects rank-ordered six different colors from the least preferred to the most preferred. The data are thus similarity data with larger numbers indicating more preferred colors and larger similarities between subjects' ideal color and actual colors. Ties were allowed, and were given the average of ranks they would have received if they were not exactly tied. The six colors used in the study are: orange (o), blue (b), grass color (g1), green (g2), red (r), and purple (p). PREFSCAL (Busing, Groenen, and Heiser, 2005) was used to analyze the data.

Figure 5 depicts the joint MDS configuration of the subjects ideal points and the six colors. The first five subjects in the data set are identified by the integers. By inspection, colors most preferred by these subjects are located close to these subject' ideal points. For example, subject 1 prefers blue and grass color, while subject 2 red and purple. Subject 5's ideal point is somewhat outlying (from the rest), indicating that another form of preference model called the vector preference model may be more appropriate for the subject (Busing et al., 2005). The analysis was run under the interval scale level assumption rather than the ordinal scale assumption, which was probably more realistic. Nonetheless, the stronger assumption was deemed preferable because of the small number of objects to avoid partially "degenerate" solutions. (Degenerate solutions are those that exhibit an excellent fit, but are substantively meaningless, e.g., all object points collapsing into one, and all ideal points into another.)

Unfolding analysis is a very useful technique in marketing research. It allows us to understand patterns of individual differences in preference judgments, and their relationships to product features and subjects' background information. This kind of analysis may eventually help marketing analysts to develop practical marketing strategies. Interested readers are referred to Takane (2007) for more examples of application of unfolding analysis.

7 A summary and software for MDS

MDS is designed for visualization of observed (dis)similarity data by distance models. In this article, we discussed essential ingredients for practical uses of MDS, such as the distance models, fitting criteria, the data collection methods, and levels of measurement scales. We also discussed several variants of MDS (simple MDS, individual differences MDS, unfolding analysis) with concrete examples of application. For more detailed discussions on these topics, see Takane et al. (2009), or any of the articles or monographs listed under "Further Reading" at the end of this article.

ALSCAL (Takane, Young, and de Leeuw, 1977) has been in SPSS (Statistical Packages for Social Sciences) for quite a long time now. However, it is slowly being superseded by a newer program PROXSCAL (Busing, Commandeur, and Heiser, 1997). The latter directly fits the distance model (rather than the squared distance model), allows multiple random starts (rather than a single rational start), and has better graphing features in the output.

Unfolding analysis has been a difficult analysis to undertake because of many instances of degenerate solutions. PREFSCAL (Busing, et al., 2005) seems to have largely overcome the problem by incorporating a penalty term in the optimization criterion. PREFSCAL has recently been incorporated into SPSS.

MULTISCALE, a maximum likelihood MDS program, can be downloaded freely at ftp://ego.psych.mcgill.ca/pub/ramsay/multiscl/ along with the program manual.

Bibliography

Borg, I., and Groenen, P. J. F. (2005). *Modern multidimensional scaling: Theory and applications.* New York, NY: Springer.

Busing, F. M. T. A., Commandeur, J. J. F., and Heiser, W. J. (1997). PROXSCAL: A multidimensional scaling program for individual differences scaling with constraints. In W. Bandilla, and F. Faulbaum (Eds.), *Softstat '97: Advances in statistical software* (pp. 237-258). Stuttgart, Germany: Lucius.

Busing, F. M. T. A., Groenen, P, J. K., and Heiser, W. J. (2005). Avoiding degeneracy in multidimensional unfolding by penalizing on the coefficient of variation. *Psychometrika, 70(1)*, 71-98.

Carroll, J. D., Chang, J. J. (1970). Individual differences and multidimensional scaling via an N-way generalization of Eckart-Young decomposition. *Psychometrika, 35(3)*, 282-319.

Coombs, C. H. (1964). *A theory of data.* New York: Wiley.

Kruskal, J. B. (1964a). Multidimensional scaling by optimizing goodness of fit to a nonmetric hypothesis. *Psychometrika, 29(1)*, 1-27.

Kruskal, J. B. (1964b). Nonmetric multidimensional scaling: A numerical method. *Psychometrika, 29(2)*, 115-129.

Ramsay, J. O. (1977). Maximum likelihood estimation in multidimensional scaling. *Psychometrika, 42(2)*, 241-266.

Ramsay, J. O. (1982). Some statistical approaches to multidimensional scaling data. *Journal of the Royal Statistical Society, Series A (General), 145(3)*, 285-312.

Shepard, R. N. (1962). Analysis of proximities: Multidimensional scaling with an unknown distance function, I and II. *Psychometrika, 27(2,3)*, 125-140 and 219-246.

Takane, Y. (2007). Applications of multidimensional scaling in psychometrics. In C. R. Rao and S. Sinharay (Eds.), *Handbook of Statistics, Vol. 26, Psychometrics*, (pp.359-400). Amsterdam: Elsevier B. V.

Takane, Y., Jung, S., and Oshima-Takane, Y. (in press, 2009). Multidimensional scaling. In R. E. Millsap and A. Maydeu-Olivares (Eds.), *Handbook of quantitative methods in psychology.* London: Sage Publications.

Takane, Y., Young, F. W., and de Leeuw, J. (1977). Nonmetric individual differences multidimensional scaling: An alternating least squares method with optimal scaling features. *Psychometrika, 42(2)*, 7-67.

Further Reading

Carroll, J. D., and Arabie, P. (1998). Multidimensional scaling. In M. H. Birnbaum (Ed.), *Handbook of perception and cognition. Vol 3: Measurement, judgment and decision making*, (pp. 179-250). San Diego, CA: Academic Press.

Cox, T. F., and Cox, M. A. A. (2001). Multidimensional scaling. Boca Raton, FL: Chapman and Hall.

Everitt, B. S., and Rabe-Hesketh, S. (1997). *The analysis of proximity data.* New York, NY: Wiley.

Kruskal, J. B., and Wish, M. (1978). *Multidimensional scaling.* Beverly Hills, CA: SAGE.

Nosofsky, R. M. (1992). Similarity scaling and cognitive process models. *Annual Review of Psychology, 43*, 25-53.

Schiffman, S. S., Reynolds, M. L., and Young, F. W. (1981). *Introduction to multidimensional scaling.* New York: Academic Press.

Young, F. W., and Hamer, R. M. (1994). *Theory and applications of multidimensional scaling.* Hillsdale, NJ: Erlbaum.

Biography

Dr. Zhidong Zhang is a post doctoral fellow in the Department of Psychology at McGill University. Dr. Zhang completed his Masters degree in Health Education and Educational Measurement at the University of Illinois at Chicago; he completed his Ph. D. in Psychometric and Statistical Methods in Educational Psychology at McGill University. Since he started his post doctoral study, his research focuses on multilevel/ hierarchical linear modeling, statistical computing, parameter estimation, model fit evaluation, and multidimensional scaling.

Dr. Yoshio Takane is a professor in the Department of Psychology at McGill University. He received his Ph. D. from the University of North Carolina at Chapel Hill. He started to contribute his wisdom in quantitative psychology and related areas more than 30 years ago. His publications have covered many theoretical and practical areas of quantitative psychology such as factor analysis, principal component analysis, multidimensional scaling, multivariate analysis, etc. He has developed many algorithms for estimating parameters in psychometric models.

Table 1: Mean dissimilarity ratings among eight sports. The eight sports are: 1. baseball, 2. basketball, 3. rugby, 4. soccer, 5. softball, 6. table tennis, 7. tennis, and 8. volleyball.

St	1	2	3	4	5	6	7
2	8.0						
3	8.7	8.3					
4	8.6	7.6	4.7				
5	1.3	9.3	9.9	9.4			
6	8.7	8.8	9.6	9.8	8.7		
7	8.0	9.1	9.4	9.6	7.9	2.1	
8	8.4	4.6	8.3	7.1	8.4	7.2	5.6

Table 2: Dissimilarity ratings among eight sports by ten subjects

	1	2	3	4	5	6	7
	7						
	8	8					
	9	9	3				
Sub. 1	2	9	10	10			
	9	9	10	9	8		
	10	10	10	10	10	5	
	10	4	9	4	9	9	8
	8						
	9	7					
	11	4	4				
Sub. 2	1	11	11	11			
	11	11	11	10	11		
	10	10	11	11	11	1	
	9	2	8	3	10	11	4
	8						
	10	10					
	8	9	4				
Sub. 3	1	9	10	9			
	10	10	10	10	10		
	10	10	10	10	9	1	
	9	6	10	10	9	10	6
	10						
	11	8					
	9	9	7				
Sub. 4	2	8	11	10			
	7	7	11	10	8		
	8	9	9	10	6	2	
	9	6	10	10	8	6	6
	7						
	8	8					
	10	5	3				
Sob. 5	1	8	9	8			
	4	7	9	6	4		
	4	7	7	8	4	2	
	6	5	7	5	6	5	4

Table 2 continued

	5						
	8	6					
	7	7	4				
Sub. 6	1	8	9	9			
	7	9	9	11	6		
	5	10	9	8	7	3	
	9	6	5	7	9	7	5
	8						
	9	7					
	9	10	8				
Sub. 7	1	9	9	9			
	9	10	9	10	10		
	8	10	10	10	7	2	
	6	6	8	8	8	7	4
	9						
	10	10					
	9	10	5				
Sub. 8	2	11	9	9			
	11	10	11	11	10		
	10	9	10	10	11	3	
	9	4	10	10	9	10	11
	10						
	8	9					
	10	7	8				
Sub. 9	1	10	11	11			
	10	10	11	10	11		
	6	9	10	11	10	1	
	8	5	8	8	10	4	5
	8						
	6	10					
	4	6	1				
Sub. 10	1	10	10	8			
	9	5	5	11	9		
	9	7	8	8	4	1	
	9	2	8	6	6	3	3

Table 3: Preference rankings among six colors by 31 subjects. Stimulus labels are: o - orange, b - blue, g1 - grass color, g2 - green, r - red, p - purple.

Sub/St	o	b	g1	g2	r	p
1	1.0	6.0	5.0	2.5	4.0	2.5
2	1.0	4.0	3.0	2.0	6.0	5.0
3	4.5	4.5	6.0	2.0	3.0	1.0
4	1.0	5.0	4.0	6.0	3.0	2.0
5	6.0	1.0	5.0	2.0	4.0	3.0
6	1.0	4.5	4.5	3.0	6.0	2.0
7	1.0	4.0	6.0	3.0	5.0	3.0
8	1.0	5.5	3.0	2.0	5.5	4.0
9	5.0	4.0	2.0	6.0	1.0	3.0
10	2.5	5.5	2.5	1.0	5.5	4.0
11	1.0	2.0	5.0	3.0	6.0	4.0
12	1.5	5.0	4.0	6.0	1.5	3.0
13	2.0	5.0	3.0	1.0	6.0	4.0
14	5.0	6.0	1.0	3.0	4.0	2.0
15	1.0	5.0	2.0	6.0	3.0	4.0
16	4.5	6.0	3.0	4.5	1.0	2.0
17	1.0	6.0	2.0	5.0	4.0	3.0
18	3.5	5.5	1.0	5.5	3.5	2.0
19	3.0	6.0	2.0	1.0	5.0	4.0
20	3.0	2.0	5.0	1.0	4.0	6.0
21	1.0	2.0	5.0	5.0	3.0	5.0
22	3.0	5.0	4.0	6.0	1.0	2.0
23	6.0	4.5	3.0	1.0	4.5	2.0
24	5.0	2.5	4.0	6.0	2.5	1.0
25	1.0	4.0	5.5	3.0	5.5	2.0
26	6.0	4.0	3.0	5.0	2.0	1.0
27	2.0	5.0	3.0	6.0	1.0	4.0
28	1.0	3.0	6.0	2.0	5.0	4.0
29	1.0	4.0	3.0	2.0	6.0	5.0
30	1.0	4.5	3.0	2.0	6.0	4.5
31	5.0	4.0	3.0	6.0	1.0	2.0

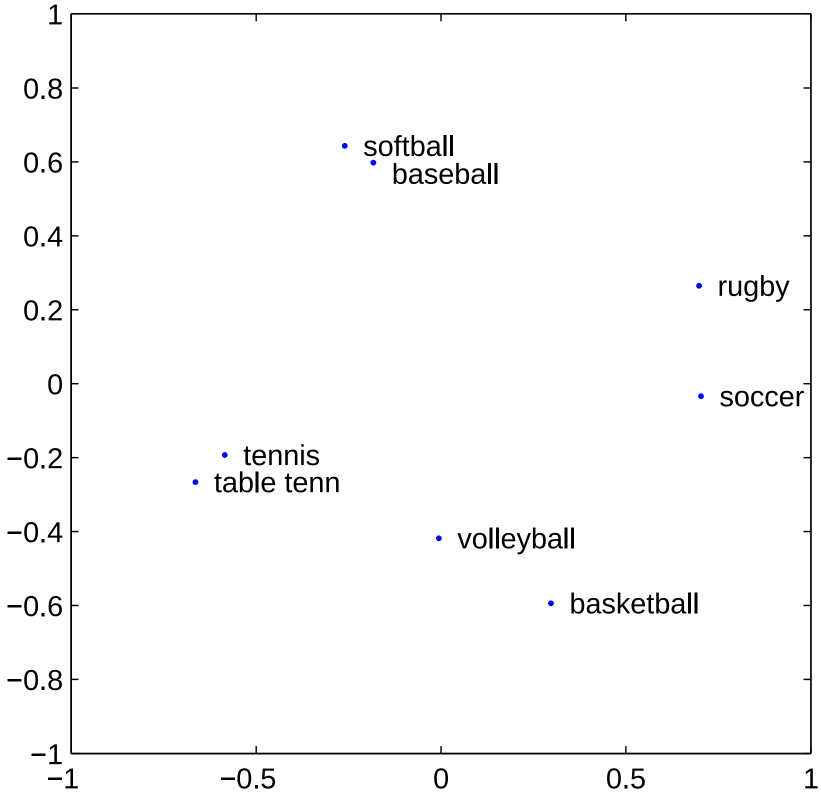

Figure 1: The two-dimensional object configuration of the eight sports from average dissimilarity ratings in Table 1. (A PROXSCAL solution obtained under the assumption of interval scaled dissimilarity data and with 100 random initial starts. Normalized raw stress = .020)

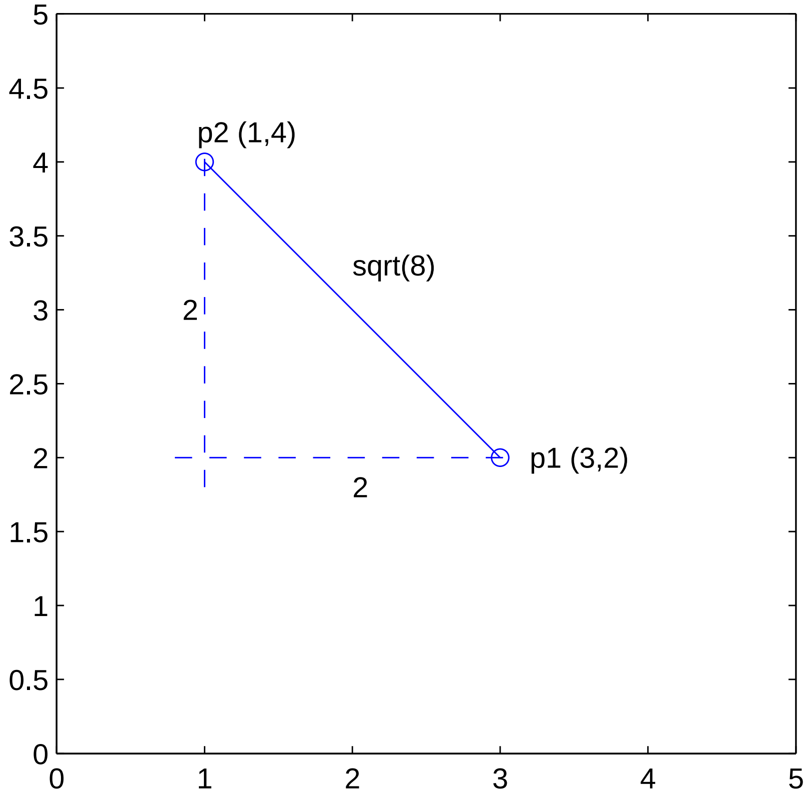

Figure 2: Calculating the Euclidean distance between two points in the two-dimensional space. (A PROXSCAL solution under the assumption of ordinal dissimilarity data and matrix conditional. Normalized raw stress = .052

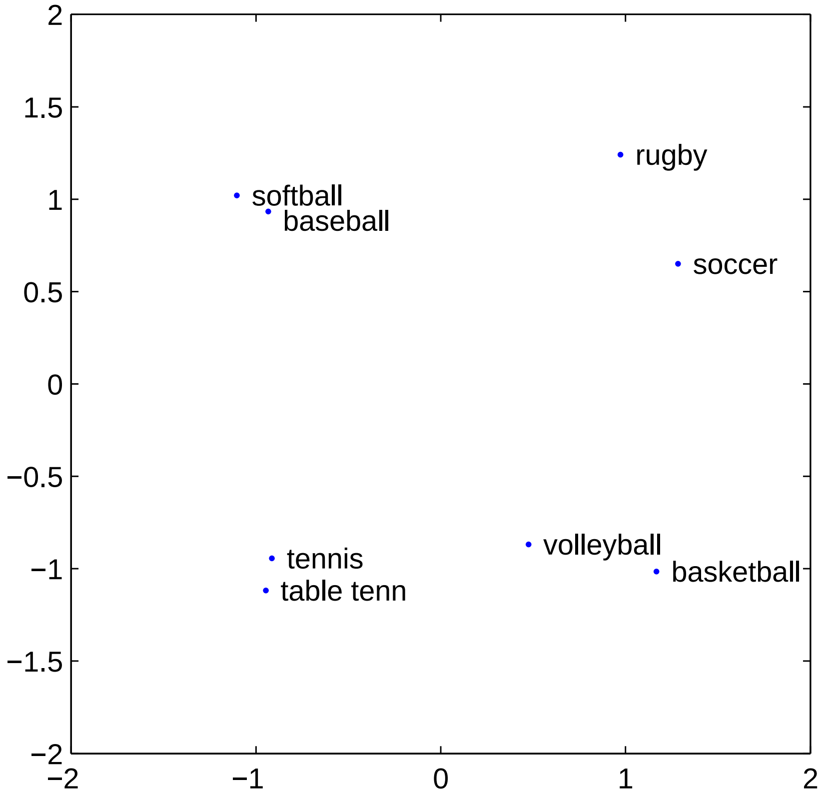

Figure 3: The two-dimensional common object configuration of the eight sports obtained by individual differences MDS of dissimilarity data in Table 2.

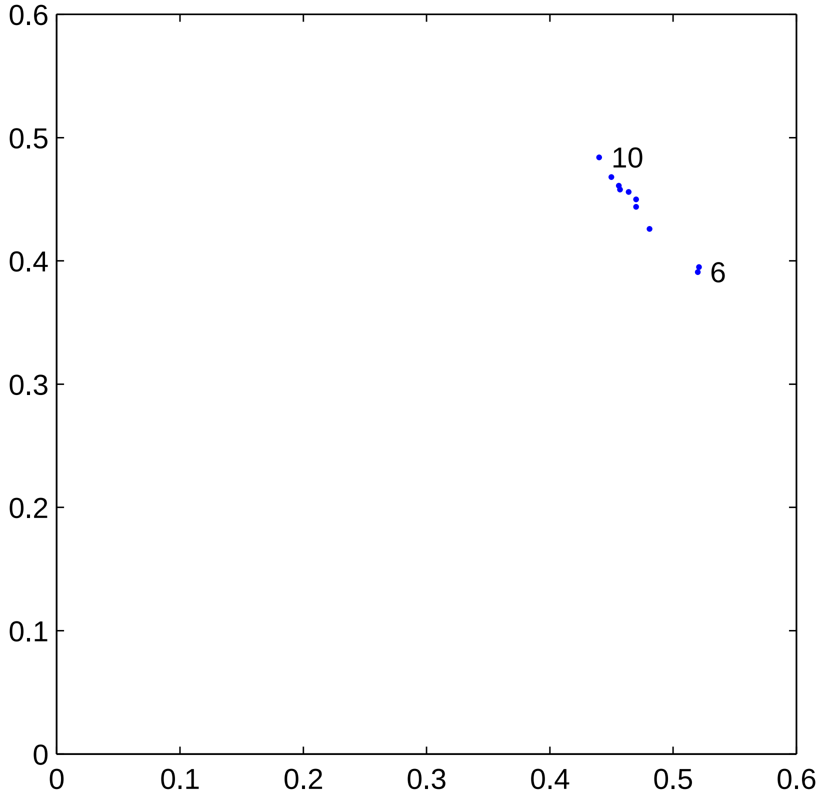

Figure 4: The plot of individual differences weights attached to the two dimensions by the ten subjects for the data in Table 3. (A PREFSCAL solution obtained under the interval scaled similarity data.)

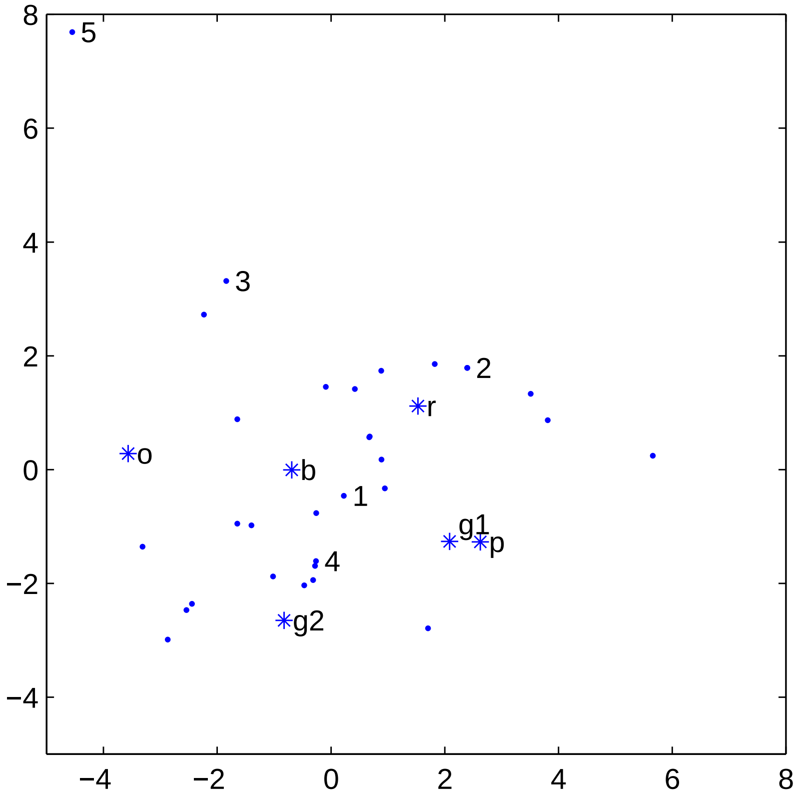

Figure 5: The joint plot of object points (Asterisks) and subjects' ideal points (dots) for the preference data on six colors presented in Table 3. The objects are labeled, and the first five subjects' ideal points are numbered while the remaining ideal points are merely indicated by dots. (Some subjects' ideal points coincide.)