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REGIONS OF SIGNIFICANCE IN MULTIPLE REGRESSION ANALYSIS

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ABSTRACT

There has been frequent confusion about the meaning of the various possible tests of significance in multiple regression, and this has led to discussions of "apparent contradictions" in regression. This paper considers the case of two predictor variables, and figures are obtained which show the regions of significance of joint regression coefficients, regression coefficients considered separately, and the multiple correlation. The intersection of these regions of significance and non-significance illustrates how the various "apparent contradictions" and anomalies may occur.

There has been frequent confusion about the meaning of the various possible tests of significance in multiple regression, and two papers in particular (Geary and Leser, 1968; and Cramer, 1972) have discussed various "apparent contradictions." Two such contradictions are particularly interesting: first, when the multiple correlation is significant but none of the individual regression coefficients are significant; and second, when all of the regression coefficients are significant but the multiple correlation is not. Cramer argues that significance tests in multiple regression should be considered in terms of model comparisons and that from the perspective of model comparisons these "apparently contradictory" cases are not contradictions. Model comparison here specifically refers to whether a certain model significantly differs from another model in its ability to predict the dependent variable. There are 2^n possible linear models when there are n predictors available. Hence, there are $2^2 = 4$ possible linear models in the two-predictor case, the simplest case of multiple regression analysis. The possible models are enumerated for the two-predictor case for further illustrative purposes, where we assume for convenience that z scores are used so that the intercepts are zero. These are

[1] $y = e_1$

[2] $y = a_1x_1 + e_2$

[3] $y = a_2x_2 + e_3$

[4] $y = b_1x_1 + b_2x_2 + e_4$

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where a_1 and a_2 are the regression coefficients for x_1 and x_2, respectively, when they are used singly; b_1 and b_2 are the regression coefficients for x_1 and x_2, respectively, when they are used jointly in one model; and the e_i represent random error. Generally, $a_1 \neq b_1$ and $a_2 \neq b_2$ unless x_1 and x_2 are uncorrelated. The correlations between y and the x_i corresponding to models (2), (3), and (4) are r_1, r_2, and R where R is a multiple correlation.

Regions of Significance and Model Comparison

It is possible to relate significance tests with model comparisons; that is, we may establish which significance test corresponds to which model comparison. There are $\binom{2^n}{2} = 2^n!/2!(2^n-2)!$ possible pairwise comparisons between models in the n-predictor case, some of which may not be of direct interest here. If we let H_0 represent some null hypothesis and H_1 some alternative hypothesis we will be interested in testing the five hypotheses below derived from models (2), (3), and (4). These are:

(I) H_0: $a_1 = 0$ (or $r_1^2 = 0$) in model (2)

vs.

H_1: $a_1 \neq 0$ (or $r_1^2 \neq 0$).

This test is equivalent to the model comparison between (1) and (2).

(II) H_0: $a_2 = 0$ (or $r_2^2 = 0$) in model (3)

vs.

H_1: $a_2 \neq 0$ (or $r_2^2 \neq 0$).

This test is equivalent to the model comparison between (1) and (3).

(III) H_0: $b_1 = 0$ in model (4)

vs.

H_1: $b_1 \neq 0$.

This test is equivalent to the model comparison between (3) and (4).

(IV) H_0: $b_2 = 0$ in model (4)

vs.

H_1: $b_2 \neq 0$.

This test is equivalent to the model comparison between (2) and (4).

(V) H_0: $R^2 = 0$ (or $b_1 = b_2 = 0$) in model (4)

vs.

H_1: $R^2 \neq 0$ (or $b_1 \neq 0$, or $b_2 \neq 0$).

This test is equivalent to the comparison between models (1) and (4). (Models (2) and (3) could provide another model comparison; but that is not of interest here.)

Since each significance test involves a distinct model comparison, there should be no confusion about the interpretation of the test results. Asking whether R^2 is significantly different from zero is quite different from asking whether one of the individual regression coefficients in a multiple regression equation is significantly different from 0. The former question relates to whether the models (1) and (4) are significantly different from each other, whereas the latter is concerned with whether models (2) and (4), or (3) and (4) are significantly different from each other.

The question arises, however, as to whether it is possible to have: (a) a significant difference between models (1) and (4) (that is, a significant R^2) when neither of the differences between (2) and (4), and (3) and (4) are significant; (b) a non-significant difference between (1) and (4) (that is, a non-significant R^2) when both (2) and (4), and (3) and (4) are significantly different. If those cases are possible, we will want to know the circumstances under which they can occur. It is in fact possible to have these cases as has been reported in a number of applications of multiple regression. It will be demonstrated below when (for what combinations of correlations) and how (for what relationship between the predictors) they come about.

Our basic strategy is to consider the tests of hypotheses I through V for the case of two predictor variables and to determine the boundaries which separate regions of significance of these hypotheses from regions of nonsignificance. Since we are dealing with z scores the regions involved are in a three dimensional space, the coordinates of which are the simple correlations among y, x_1, and x_2. In order to provide a two dimensional representation it is convenient to fix the correlation between the predictor variables at each of several values r and to determine the regions as a function of the correlations, r_1 and r_2, between the predictor variables and the criterion variable.

There are five boundaries in the space, each corresponding to one of the five significance tests I through V. Each boundary divides the space into two distinct regions, one region corresponding to the significance, the other to the non-significance of the coefficient tested. These regions are shown in Figures 1 through 3, for

$r = .5$, $r = .9$ and $r = 0$, a .05 α level, and a sample size of 23 giving 20 degrees of freedom for error in the test of R^2:

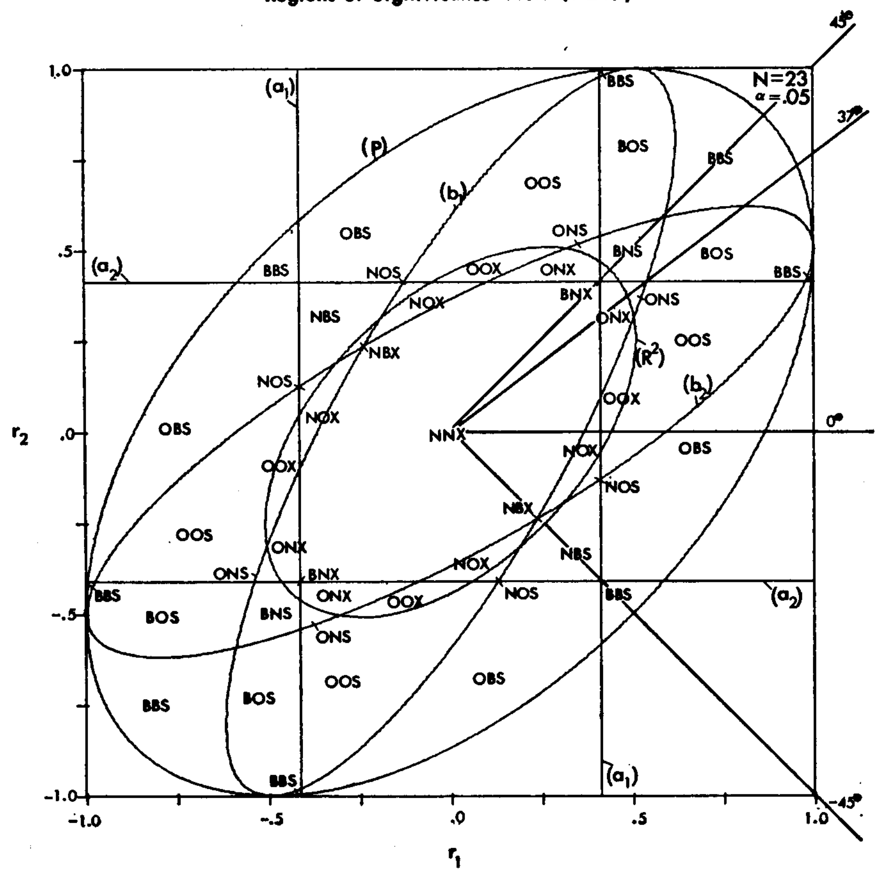
(1) The boundary for the significance test of a_1 is determined by $(r_1^2/1)/[(1-r_1^2)/(N-2)] = F_\alpha(1,N-2)$ where N is the number of observations, and F_α is the critical value of the F distribution associated with the α significance level. The set of solutions of this equation forms two vertical lines in the two dimensional space, since a_1 is independent of r_2 and r. In the region between the lines denoted by (a_1), a_1 is not significant.

(2) The boundary for the significance test of a_2 is similar to that for a_1 consisting of two horizontal lines denoted by (a_2).

(3) The boundary for the significance test of b_1 is determined by

FIGURE 1

Regions of Significance Tests ($r = .5$)



$[(R^2-r_2^2)/1]/[(1-R^2)/(N-3)] = F_\alpha(1,N-3)$. For every value of r this equation forms an ellipse and combinations of r_1 and r_2 outside this ellipse lead to significant values of b_1. The ellipse is denoted by (b_1).

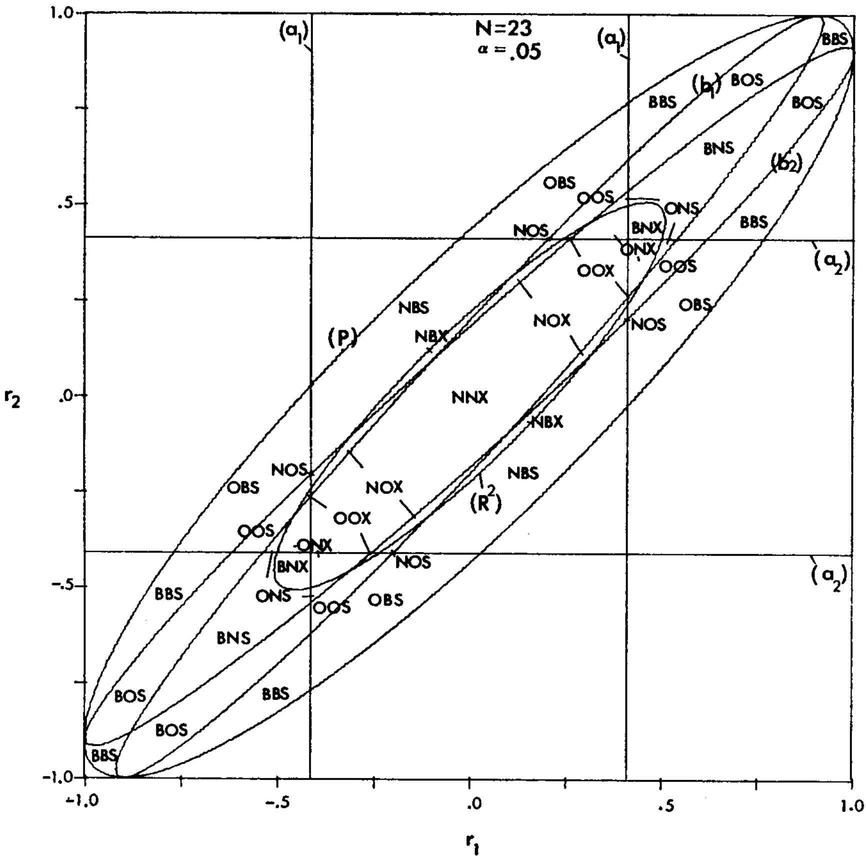
(4) The boundary for the significance test of b_2 is also an ellipse and is denoted by (b_2).

(5) The boundary for the significance test of R^2 is determined by $(R_2/2)/[(1-R^2)/(N-3)] = F_\alpha(2,N-3)$. This region is an ellipse denoted by (R^2).

Not all combinations of r, r_1, and r_2 can occur simultaneously, and the interior of the large ellipse labeled (P) in each figure is

FIGURE 2

Regions of Significance Tests (r = .9)

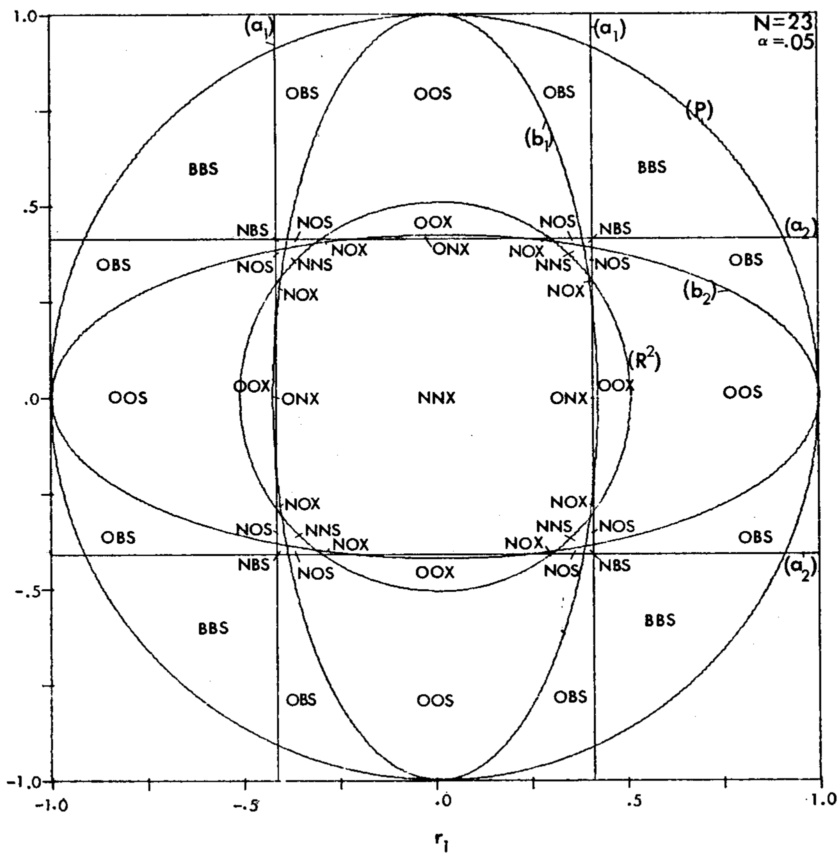


the locus of all possible sets of values. The ellipse (R^2) has the same orientation as (P). Both have their major axes along the 45° line and become straight lines as r approaches one; for $r = 0$ they are concentric circles. The ellipses for b_1 and b_2 are always true ellipses. The major axis of the ellipse for b_1 is oriented vertically when $r = 0$ and approaches the 45° line as r goes to one. The ellipse becomes thinner as r gets larger. The ellipse for b_2 is similar, except that its major axis is oriented horizontally when $r = 0$ and approaches the 45° line as r goes to one.

There are a number of subregions which must be null. Cramer (1972) proved that if both a_1 and b_2 are significant, R^2 must also

FIGURE 3

Regions of Significance Tests (r=.0)



be significant. We may interpret this by saying that if a given variable is significant and a second adds significantly to it, the combination must be significant.

THE SIGNIFICANCE REGIONS

The intersections of the regions in the figures define subregions which may be labeled according to the significance of the a_i, the b_i and R^2. Because of the symmetry it is convenient to label these regions with a three letter code where the first letter represents the a_i, the second represents the b_i, and the third represents R^2. The a_i may be both significant (B), have only one significant (O), or have neither significant (N), and the same applies to b_i. R^2 may be significant (S) or not significant (X). The code BOS then means that both of the a_i are significant, one of the b_i is significant, and R^2 is significant. Any subregion for which a_1 and b_2 are significant but R^2 is not, must be null. This implies that OBX, BOX, and BBX are null. Examination of the figures suggests an additional impossible case which, so far as we know, has not been noted before. If a_1 and b_2 are significant, either a_2 or b_1 must be significant. To prove this we note that if a_1 is significant and a_2 is not, r_1^2 is greater than r_2^2. This implies that $R^2-r_2^2$ is greater than $R^2-r_1^2$. Since these are the numerators of the F statistics for b_1 and b_2 while both have the same denominators, the significance of b_2 implies the significance of b_1. On the other hand, if b_2 is significant and b_1 is not, $R^2-r_1^2$ is greater than $R^2-r_2^2$, implying that r_2^2 is greater than r_1^2. If then a_1 is significant, a_2 must be also, concluding the proof. We note that for the cases OOS or OOX where one of the a_i and one of the b_i are significant, the significant coefficients must be either a_1 and b_1 or a_2 and b_2.

The 15 non-null regions are listed in Table 1. Certain of these regions are null for some values of r as indicated in the table. The symmetry in the figures is readily apparent and comes from two sources. There is symmetry with respect to the 45 degree line reflecting the labeling of axes as r_1 and r_2. There is also symmetry with respect to the -45 degree line resulting from changing the signs of both r_1 and r_2 which is equivalent to multiplying y by -1. There are then four equivalent segments of which we need consider only one bounded by the 45 degree radial and the -45 degree radial with r_1 positive.

To more closely examine the significance patterns, let us ex-

Table 1
Possible combinations of significance of the a_i, b_i, and R^2

a_i	b_i	R^2	
B	B	S	
B	O	S	impossible for r near zero
B	N	S	impossible for r near zero
B	N	X	impossible for r near zero
O	B	S	
O	O	S	if a_k is significant b_k is also
O	O	X	if a_k is significant b_k is also
O	N	S	impossible for r near zero
O	N	X	
N	B	S	
N	B	X	impossible for r near zero
N	O	S	
N	O	X	
N	N	S	impossible except for r near zero
N	N	X	

B = both significant
O = one significant
N = none significant
S = significant
X = not significant

amine Figure 1 which illustrates moderately correlated independent variables (r = .5). We can observe that all regions except NNS are represented in this segment of the figure. By moving along four lines from the point ($r_1 = 0$, $r_2 = 0$), we encounter almost all of the significance regions. Moving along the 45 degree line where $r_1 = r_2$, we go from the NNX region where nothing is significant, through BNX at r_1 of about .4 where only the separate coefficients are significant. We then pass through BNS centered at about $r_1 = .5$ which is one of Geary and Leser's apparent contradictions with neither joint regression coefficient significant, but the multiple correlation significant. At $r_1 = .6$ we reach region BBS where everything is significant. The sequence of events is first the separate coefficients become significant, then the multiple correlation, and finally the joint coefficients. Along the 37 degree line the pattern is similar except that first a single coefficient becomes significant (ONX), then the multiple correlation (ONS), then the second separate coefficient and one joint coefficient (BOS) and finally all coefficients. Along the zero degree line where r_1 = 0 a joint coefficient becomes significant first (NOX). A separate coefficient becomes significant next (OOX) and then the multiple correlation (OOS). There is a large region in which both joint coefficients are significant (OBS) but of course the other separate coefficients cannot be significant along this line. Finally along the -45 degree line where $r_2 = -r_1$, the joint coefficients reach significance first (NBX) followed by the multiple correlation (NBS) and then all coefficients (BBS). The only regions not encountered along these lines are (NOS) at ($r_1 = .4$, $r_2 = -.1$) and an additional tiny (BBS) region near ($r_1 = 1.0, r_2 = .4$).

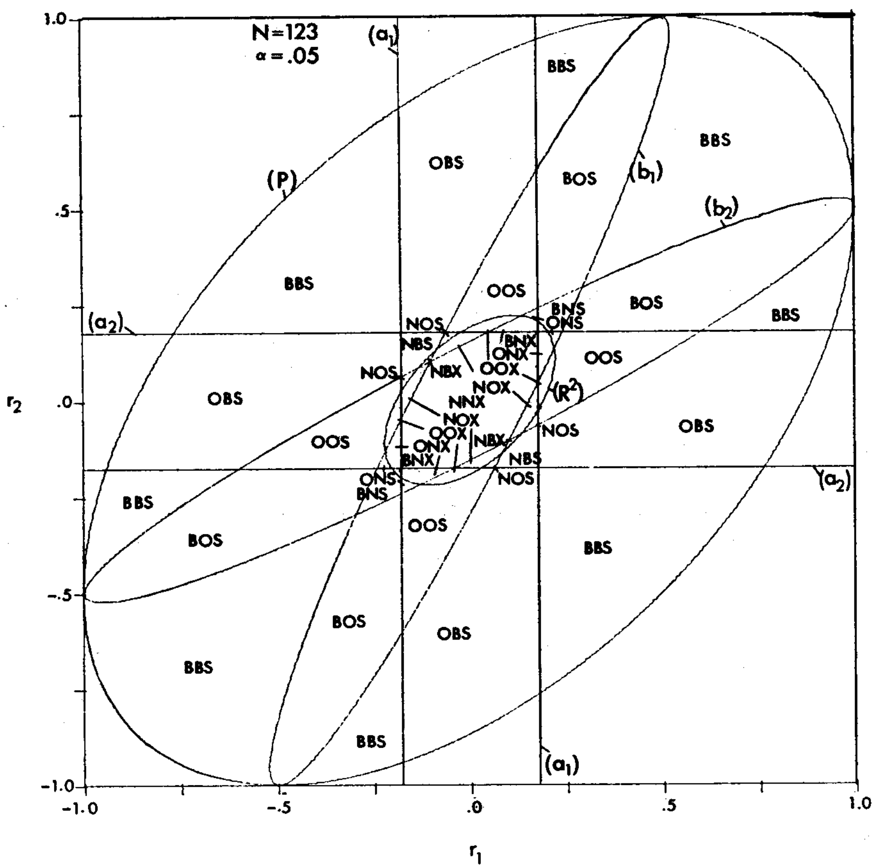
We find then that the most common way to obtain significance of all coefficients (BBS) in the case of moderately correlated independent variables is obviously for both to be highly positively (or negatively) correlated with the dependent variable (correlations of .6 or higher). Significance can be obtained with moderate correlations of opposite sign (.4 to .5). One of Geary and Leser's apparent contradictions where the multiple correlation is significant but neither of the joint coefficients are (BNS and ONS) occurs for correlations of about .5 with the same sign. On the other hand there is only a very small region where both joint coefficients are significant and the multiple correlation is not (NBX). This occurs for correlations of opposite sign, with magnitudes of about .25. The opposite effect, both separate coefficients significant and the multiple correlation not significant, occurs only in a small region with correlations of the same sign (about .45).

Figure 2 illustrates the significance patterns for the case of a high correlation between the independent variables ($r = .9$). The effect of the correlation is to decrease the size of the minor axes of the ellipses and to make the slopes of the ellipses for the joint regression coefficients approach 45 degrees. As the correlation r goes to 1, the ellipses approach the 45 degree line. The general pattern is similar to Figure 1 except that both the NBS and BNS regions are somewhat enlarged.

The case of uncorrelated independent variables in Figure 3 provides a more predictable pattern. It should be noted that although the regression sum of squares corresponding to a single regression coefficient is the same for that variable considered either separately or in combination with another variable, the error sum of squares and degrees of freedom are different in the two cases. The symmetry in Figure 3 is such that only the first quadrant between 0 and 45 degrees need be considered. The only anomaly is the small (NNS) region for correlations of any sign in the neighborhood of .35. The multiple correlation is significant although none of the individual coefficients are significant; this can only occur with independent variables whose correlation is near zero. The progression of regions is just what one would expect; along the 45 degree line the joint coefficients become significant first because of the reduced error variance, while along the zero degree line the behavior of the two types of coefficients is essentially the same.

FIGURE 4

Regions of Significance Tests (r=.5)



In order to examine the effect of a change in sample size on the significance regions, we have also obtained figures for a sample size of 123. The only effect of an increase in sample size is to reduce the size of the regions, as shown in Figure 4 for $r = .5$. This reflects the fact that smaller correlations are sufficient to provide significance with increasing sample size. The orientations and relations of the ellipses remain unchanged.

Summary

In this paper we have described the patterns of significance in multiple regression in terms of correlations between variables. This was done by drawing a map of significance regions corresponding to the significance tests for each of several fixed values of the correlation between the two predictors. We have found some interesting patterns and located them in the figures. It is hoped that the geometric representation can serve as a guide for better understanding of the relations among variables in regression analysis.

References

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