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CONSTRAINED DEDICOM

HENK A. L. KIERS

UNIVERSITY OF GRONINGEN

YOSHIO TAKANE

MCGILL UNIVERSITY

The DEDICOM method for the analysis of asymmetric data tables gives representations that are identified only up to a nonsingular transformation. To identify solutions it is proposed to impose subspace constraints on the stimulus coefficients. Most attention is paid to the case where different subspace constraints are imposed on different dimensions. The procedures are discussed both for the case where the complete table is fitted, and for cases where only off-diagonal elements are fitted. The case where the data table is skew-symmetric is treated separately as well.

Key words: asymmetric relationships, alternating least squares.

Many research questions lead to the analysis of a square data table consisting of relationship measures among a set of n objects. Often these relationships denote similarities or distances among the objects, and usually such measures are symmetric, that is, the similarity between objects i and j is the same as that between j and i. However, relationships among a set of objects need not be symmetric. Asymmetric relationship data may, for example, concern friendships that need not be mutual, mobility tables, import/export figures, or confusion frequencies for pairs of consonants. Many methods have been proposed for the analysis of asymmetric data (see, e.g., Chino, 1991, for a review). An interesting method for the analysis of such square, asymmetric relationship matrices is DEDICOM, which is an abbreviation of DEcomposition into DIrectional COMponents (Harshman, 1978; also, see Harshman, Green, Wind, & Lundy, 1982).

In DEDICOM the asymmetric data table X ($n \times n$) is modeled by

$$X = ARA' + E, \tag{1}$$

where A is an $n \times r$ matrix of coefficients expressing to what extent an object takes part in each of the r "idealized" objects, R is an asymmetric $r \times r$ matrix of relationships between idealized objects, and E is an $n \times n$ matrix of error terms. Each idealized object should be seen as an object with a particular basic aspect of the relations among the objects. According to the structural part of the model, the relationship between objects i and j can be written as $\sum_k \sum_l a_{ik}a_{jl}r_{kl}$. That is, it is decomposed as the sum of asymmetric relationships between idealized objects, multiplied by the coefficients that express to what extent an object takes part in the idealized objects.

Harshman (1978) proposed fitting the model to the data by minimizing

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Requests for reprints should be sent to Henk A. L. Kiers, Department of Psychology (P/OP), Grote Kruisstr. 2/1, 9712 TS Groningen, THE NETHERLANDS.

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$$\sigma(A, R) = \|X - ARA'\|^2, \tag{2}$$

over A and R, and provided an algorithm for it (Harshman, 1981a). Harshman's algorithm was based on treating the left-hand A and the right-hand A as if they are independent, without guarantee that they are equal upon convergence. Monotonically convergent algorithms where left-hand A and right-hand A are not distinguished have been proposed by Kiers (1989) and Kiers, ten Berge, Takane, and de Leeuw (1990). For convenience, matrix A is usually constrained to be columnwise orthonormal, as can be done without loss of generality. Even with this constraint, the solution is not identified. One way of identifying it is to rotate the matrix A to simple structure, for instance, by varimax (e.g., Harshman et al., 1982). Many other orthogonal simple structure rotations exist, and it seems difficult to decide which of these should be used. Moreover, there seems to be no compelling reason for constraining the columns of A to be mutually orthogonal, hence, oblique simple structure rotations may be considered as well. Note that whatever the rotation is, the rotation T should be compensated for in R; that is, if A is replaced by $\tilde{A} = AT$, then R should be replaced by $\tilde{R} = T^{-1}R(T')^{-1}$. Rather than trying to solve the problem of choosing the most appropriate simple structure rotation, in the present paper we will propose a method to invoke a particular, usually simple, structure in the solution by imposing certain constraints on matrix A. Before discussing how this can be done, we will pay a little more attention to the interpretation of a DEDICOM solution, illustrated by an artificial example.

To interpret a DEDICOM solution it is crucial to have an idea of what the idealized objects stand for. A simple situation is sketched in the following example. Suppose there are 6 car types that belong to two groups, say the first three types of cars (A, B, and C) are small cars, and the last three (U, V, and W) are big cars. Suppose, further, there is a data matrix X consisting of car-switching frequencies, with x_{ij} denoting the number of times a car of type i is replaced by a car of type j. Obviously, such a data table is likely to be asymmetric, because it is not to be expected that type i is replaced by type j just as often as type j is replaced by type i. Now an ideal two-dimensional DEDICOM representation of such a data table would be one in which the first column of A has nonzero elements only in the first three rows (corresponding to cars A, B, and C) and the second column of A has nonzero elements only in the last three rows (corresponding to U, V, and W). Assuming that the nonzero elements are not very different, one may interpret the first "idealized object" as "the typical small car", and the second as "the typical big car". Then the elements of matrix R represent the asymmetric relationships among typical small cars and typical big cars. Specifically, if the DEDICOM representation describes the data table perfectly, and the nonzero elements of A are positive and are scaled to unit sums, the element r_{11} represents the frequency of small cars being replaced by small cars, r_{12} the frequency of small cars being replaced by big cars, etc. To see this, consider the element x_{AU}, which can be reconstructed by the model parameters as the product $a_{A1}a_{U2}r_{12}$. Summing all elements of X for which the row element is a small car, and the column element is a big car, one finds

$$\textstyle\sum_{i\in(A,B,C)} \sum_{j\in(U,V,W)} a_{i1}a_{j2}r_{12} = (\sum_{i\in(A,B,C)} a_{i1})(\sum_{j\in(U,V,W)} a_{j2})\, r_{12} = r_{12},$$

using the fact that $(\sum_{i\in(A,B,C)} a_{i1}) = (\sum_{j\in(U,V,W)} a_{j2}) = 1$.

The above example is unrealistic in at least two respects. Firstly, we assume that the DEDICOM representation is perfect, and secondly, we assume that the solution will give a matrix A with only one nonzero element per row. In practice, the representation will usually not be perfect, and matrix A will usually not be as simple as in the example. One way to approximate the ideal situation of the above example is, first to find a DEDICOM representation that approximates the data matrix sufficiently well (as is done by minimizing σ over matrices A and R for different dimensionalities), and secondly to transform the solution of A in such a way that it optimally resembles the ideal matrix A from the example (which is in fact aimed at by simple structure rotations). As mentioned above, we end up with the problem of deciding how to rotate A, that is, which simple structure criterion should be used. Apart from the arbitrariness of the choice of a rotation procedure, there is some arbitrariness caused by the approximation: It is possible that a different representation that approximates the data only slightly worse will give a solution that is easier to interpret, giving a simpler description of "idealized objects". For instance, it is possible that, if the optimal representation is based on a matrix A with a first column $(.8\ .8\ .7\ .2\ .2\ .2)'$, then replacing this column by the perfectly simple $(.8\ .8\ .7\ 0\ 0\ 0)'$ would affect the optimality of the representation only marginally. In such cases we would prefer the latter simple solution over the negligibly better optimal one. To invoke such a solution is especially useful in cases where there is external information on the objects (like the information that cars A, B, and C are small, and U, V, and W are big). Such solutions can be obtained by constraining matrix A to have the particular form desired. Note that the particular choice of the constraint depends on the external information available. Often, such information can be represented by subspace constraints (see Takane & Shibayama, 1991, and Takane, Kiers, & de Leeuw, 1991, for examples of this in a different context). Whatever the form of the constraint is, it will generally affect the optimality of the solution. Obviously, a prerequisite of using constrained solutions is that they should still represent the data sufficiently well.

The main purpose of the present paper is to propose methods for constraining matrix A such that interpretation is facilitated and the DEDICOM representation of the data is, given the constraints, as good as possible. To start with, we will discuss fitting the DEDICOM model subject to the constraint where A is forced to be in a specified column space, which seems to be the most commonly used constraint. Next, DEDICOM with more specific kinds of constraints will be discussed in which each dimension of A will be constrained in a different way (compare Takane et al., 1991). Algorithms for these constrained minimization procedures will be presented, uniqueness of the solution will be discussed, and the methods will be illustrated on an example data set.

After discussing constraints on the standard DEDICOM procedure, we will treat two special cases. First, we will discuss constrained variants of off-diagonal DEDICOM, which denotes the variant of DEDICOM in which only off-diagonal elements of X are fitted. This model is of interest in situations where the diagonal elements of X are not meaningful, or of an essentially different nature than the off-diagonal elements. Secondly, we will discuss the case where X is skew-symmetric. The special form of the DEDICOM solution for skew-symmetric matrices motivates the use of a special class of constraints. To start, however, we will discuss the case where general subspace constraints are imposed on the DEDICOM parameters A.

DEDICOM with Subspace Constraints

The first type of constraint on the coefficients matrix A to be discussed here is that the columns of A should lie in a prescribed column space. If G $(n \times m;\ m < n)$ is an orthonormal basis for that column space, the constraint implies that A can be written as $A = GU$ for a certain $(m \times r)$ matrix U. Note that any constraint written as $A = HV$ for some fixed matrix H can be rewritten as $A = GU$ by defining G, for example, as the Gram-Schmidt orthonormalized version of H, such that $H = GT$ for a certain matrix T, and hence $U = TV$. Substituting the expression $A = GU$ for A in the DEDICOM loss function, we find

$$\sigma(A, R) = \|X - ARA'\|^2 = \|X - GURU'G'\|^2 = \sigma_c(U, R). \tag{3}$$

Using the columnwise orthonormality of G, (3) can be rewritten as

$$\begin{aligned}\sigma_c(U, R) &= \|X - GURU'G'\|^2 \\ &= \|X - GG'XGG'\|^2 + \|G'XG - URU'\|^2,\end{aligned} \tag{4}$$

(see Carroll, Pruzansky, & Kruskal, 1980, p. 7). Clearly, the first term in the right-hand-side of (4) is constant over U and R, and minimizing (3) is equivalent to minimizing $\|G'XG - URU'\|^2$ over U and R. The latter problem is the original DEDICOM minimization problem applied to $G'XG$ rather than X. Hence, to find U and R the DEDICOM model is fitted to the matrix $G'XG$. Having found the optimal U one can find A by $A = GU$.

Some notes are in order here. Firstly, it should be noted that, if the subspace constraint is given as $A = GU$ for some nonorthonormal matrix G, then one should first replace G by an orthonormal basis of the columns of G, and next proceed as above. Secondly, it should be noted that, having obtained the constrained solution, one still has not completely identified A. One may still apply nonsingular $r \times r$ transformation matrices to A and compensate for this by replacing R by $T^{-1}R(T')^{-1}$. For this reason it is questionable if the subspace constraint will really facilitate interpretation considerably. In cases where more than simple subspace information is available, one may proceed differently, as exemplified below.

DEDICOM With Different Constraints on Different Dimensions

Instead of imposing a subspace constraint on the complete matrix A, one may use external information in a more specific way by imposing different subspace constraints on different dimensions (Takane et al., 1991). For instance, in the car switching example discussed in the introduction, one might want to impose a constraint on the first dimension such that it produces zero values for the last three car types, and a different constraint on the second dimension to invoke zero values for the first three car types. Specifically, one might require that $\mathbf{a}_1$ and $\mathbf{a}_2$ (the first and second columns of A) be in the subspaces spanned by

$$G_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } G_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. Constraints other than those producing a simple structure can be used, of course. In general, one may impose the constraint that $\mathbf{a}_l = G_l\mathbf{u}_l$, where G_l is a columnwise orthonormal basis of a prescribed subspace, $l = 1, \ldots, r$. Note that, in general, $A'A = I_r$ may no longer be assumed. Instead we will usually assume Diag $(A'A) = I_r$, which can be done without loss of generality. An alternating least squares algorithm for minimizing (2) subject to $\mathbf{a}_l = G_l\mathbf{u}_l$, and $\mathbf{a}_l'\mathbf{a}_l = 1$, $l = 1, \ldots, r$, is derived in the Appendix. In the next sections, we will discuss some conditions for uniqueness of solutions of DEDICOM subject to different constraints on different dimensions, and apply the method to an example data set.

Uniqueness Conditions

As observed above, the unconstrained DEDICOM solution is not unique. Any nonsingular transformation of A can be compensated for by a transformation of R. The same holds for general subspace constraints, discussed in the second section. In the dimensionwise constrained variants of DEDICOM there still is a nonuniqueness of scale and reflection, and one may arbitrarily impose the constraint that A has unit column sums of squares. However, apart from this scaling and reflection, the solution is typically unique. In fact, we have the following sufficient condition for uniqueness of the DEDICOM solution when the dimensions are constrained by different subspaces.

Theorem. If the columns of A are constrained to be in subspaces spanned by matrices $G_1, \ldots, G_r$, then given $\hat{X} = ARA'$ for full rank matrices A and R, matrix A is unique up to scaling and reflection of the columns of A if and only if the subspaces spanned by G_i and by the columns $\mathbf{a}_l$ $(l \neq i)$ of A are disjoint for all i.

Proof. Suppose that for each i, the subspaces spanned by G_i and by columns $\mathbf{a}_l$ $(l \neq i)$ are disjoint. We want to prove that, then, $ARA' = A^*R^*A^{*\prime}$ implies $A^* = AD$, for some diagonal matrix D. From $ARA' = A^*R^*A^{*\prime}$ it follows at once that $A^* = AT$ for some nonsingular T. We have to prove that there is no nondiagonal transformation matrix T such that AT satisfies the same subspace constraints as A does. Let $\mathbf{t}_i$ denote the i-th column of T, $i = 1, \ldots, r$. To satisfy the subspace constraints for the columns of $A^* = AT$, we must have $A\mathbf{t}_i = G_i\mathbf{v}_i$, for a certain vector $\mathbf{v}_i$, $i = 1, \ldots, r$. Then $A\mathbf{t}_i = t_{1i}\mathbf{a}_1 + \cdots + t_{ri}\mathbf{a}_r = t_{ii}\mathbf{a}_i + (\mathbf{a}_1 \vdots \ldots \vdots \mathbf{a}_{i-1} \vdots \mathbf{a}_{i+1} \vdots \ldots \vdots \mathbf{a}_r)\mathbf{t}_{-i} = G_i\mathbf{v}_i$, where $\mathbf{t}_{-i}$ is the vector with the same elements as $\mathbf{t}_i$, except the i-th. The disjointness of the subspaces spanned by $(\mathbf{a}_1 \vdots \ldots \vdots \mathbf{a}_{i-1} \vdots \mathbf{a}_{i+1} \vdots \ldots \vdots \mathbf{a}_r)$ and G_i implies that $\mathbf{t}_{-i} = 0$, for all i, hence T is diagonal, which had to be proven.

Conversely, if the spaces spanned by G_i and $(\mathbf{a}_1 \vdots \ldots \vdots \mathbf{a}_{i-1} \vdots \mathbf{a}_{i+1} \vdots \ldots \vdots \mathbf{a}_r)$ are not disjoint, there is a linear combination $\mathbf{b}_i$ of columns $\mathbf{a}_l$ $(l \neq i)$ that is in the subspace spanned by G_i. Then a matrix A^* defined as the matrix with the same columns as A, except the i-th, which is replaced by $\mathbf{a}_i + \mathbf{b}_i$, obviously satisfies the constraint that $\mathbf{a}_i$ be in the column space of G_i. This A^* is computed from A by a nonsingular transformation T that is defined as an identity matrix except for the elements in column i of T. If $R^* \equiv T^{-1}R(T^{-1})'$, we have $ARA' = A^*R^*A^{*\prime}$. Hence, we find the same $\hat{X}$ for matrices A and A^* that differ by more than a columnwise scaling. This proves that uniqueness of A implies that no linear combination of the columns $\mathbf{a}_l$ $(l \neq i)$ is in the subspace spanned by G_i, for every i; that is, the spaces spanned by G_i and $(\mathbf{a}_1 \vdots \ldots \vdots \mathbf{a}_{i-1} \vdots \mathbf{a}_{i+1} \vdots \ldots \vdots \mathbf{a}_r)$ are disjoint. □

The above theorem is useful in establishing uniqueness of a solution *after* having obtained it. The following corollary gives a sufficient condition for uniqueness that can be evaluated *before* finding the solution for A.

Corollary. If the columns of A are constrained to be in subspaces with bases $G_1, \ldots, G_r$ and if $(G_1 \vdots \ldots \vdots G_r)$ has full column rank, then, given $\hat{X} = ARA'$, matrix A is unique up to scaling and reflection of the columns of A.

Proof. If $(G_1 \vdots \ldots \vdots G_r)$ has full column rank, for every i and every set of vectors $\mathbf{u}_l$, the subspaces spanned by G_i and $\mathbf{a}_l = G_l\mathbf{u}_l$ $(l \neq i)$ are disjoint. It follows from the above derived theorem that this implies uniqueness of the solution. □

That this condition is not necessary for uniqueness of A to hold can be seen from the following example. If G_1 and G_2 have one column in common (and hence $(G_1 \vdots G_2)$ is not of full column rank), then the solution for A can still be unique. For example, suppose

$$G_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Any other candidate solution A^* for A should at least be in the same column space of A (since $ARA' = A^*R^*A^{*\prime}$). Hence, $A^* = AT$ for a certain matrix T. Any nondiagonal T would produce an $A^* = AT$ with either a_{31}, or a_{22}, or both nonzero. Hence, the columns of A^* would not both satisfy the constraints that they be in the subspaces spanned by G_1 and G_2, respectively. Therefore, in this case T must be diagonal, and A is unique up to scaling and reflection of its columns.

Exemplary Analysis of Car Switching Data

Harshman et al. (1982, p. 221) provided a data set on car switching frequencies among 16 types of cars, ranging from Subcompact/Domestic to Luxury Import. The abbreviations used here (and in Harshman et al., 1982) consist of two components: The first three characters mainly indicate size (SUB = subcompact, SMA = small specialty, COM = compact, MID = midsize, STD = standard, and LUX = luxury); the fourth character indicates mainly origin or price (D = domestic, C = captive imports, I = imports, L = low price, M = medium price, S = specialty). The authors produced an unconstrained four-dimensional solution, but at that time lacked a procedure for finding an optimal solution (see p. 237). Using the currently available algorithm we did obtain the four-dimensional solution (which accounted for 92.0% of the total sum of squares), but, after normalized varimax rotation of the orthonormal matrix A, we found one dimension to be related mainly to the single category of medium priced standard cars. DEDICOM with only three dimensions still accounted for 86.4% of the total sum of squares. For these reasons the three-dimensional solution was preferred. In Table 1 we report the optimal normalized varimax rotated solution for A (with the columns of A taken to be orthonormal). The first dimension of A pertains to a cluster of plain large and midsize cars, the second dimension mainly represents fancy large cars, and the third mainly represents the small/specialty cars. In fact, we found more or less the same dimensions as Harshman et al., except that specialty and small cars have been merged into one dimension.

These results suggest that the solution could be simplified into one in which each car category is represented by one dimension only. For that purpose we constrained the columns of A such that the first dimension has nonzero elements only for COML, COMM, MIDD, MIDI, and STDL, the second dimension only for STDM, LUXD and LUXI, and the third dimension only for SUBD, SUBC, SUBI, SMAD, SMAC, SMAI, COMI, and MIDS. The DEDICOM solution thus constrained accounted for 83.7% of the total sum of squares. We now normalized A to unit column sums (instead of sums of squares), because then the resulting matrix R represents approximations to the switching frequencies between the nonoverlapping clusters of cars, as explained in the introduction. The resulting matrices A and R are given in Table 1 under the heading constrained nonoverlapping solution. Note that the numbers are not comparable in size to those of the previous solution. The dimensions have almost the same interpretation as before, except that the relative importances of the car types for their clusters have

TABLE 1

(Constrained) DEDICOM Applied to the Car Switching Data

	unconstrained followed by varimax			constrained nonoverlapping			constrained overlapping		
	1	2	3	1	2	3	1	2	3
matrix A									
SUBD	.13	-.02	.36			.21			.43
SUBC	.02	.00	.03			.02			.04
SUBI	.03	.01	.30			.14			.29
SMAD	.01	.03	.53			.22			.46
SMAC	.00	.00	.00			.00			.00
SMAI	.00	.01	.09			.03			.08
COML	.24	-.11	.17	.17			.27		
COMM	.10	-.01	.06	.08			.13		
COMI	.02	.00	.03			.02			.04
MIDD	.54	.00	.12	.39			.61		
MIDI	.02	.00	.02	.02			.03		
MIDS	.09	.24	.58			.36		.21	.66
STDL	.68	-.08	-.18	.34			.54		
STDM	.32	.67	-.27		.68		.48	.71	-.28
LUXD	-.23	.69	.05		.31		-.07	.68	
LUXI	.00	.02	.01		.01			.02	
matrix R (divided by 1000)									
dim. 1	127	57	78	325	108	362	127	30	106
dim. 2	26	92	23	66	162	107	-6	83	9
dim. 3	17	12	75	94	56	341	35	17	86
fit		86.4 %			83.7 %			85.3 %	

changed slightly. Especially the relative importance of STDM and LUXD for the second cluster has changed. Matrix R now reveals that, apart from many within cluster switches, there is a very high frequency of switches from the first (plain large and midsize cars) cluster to the third (small/specialty cars) cluster.

The nonnegligible change in relative importances of STDM and LUXD, as well as other major differences between the unconstrained and constrained solutions may have come about because certain relatively high "loadings" in the unconstrained solution were constrained to zero in the constrained solution. To see what happens if such elements in A are left unconstrained we did a third analysis in which the same elements were constrained to zero, except those that were larger than .20 or smaller than −.20 in the original unconstrained varimax solution. Specifically, we no longer fixed to zero the loadings of MIDS on the second dimension, of STDM on the first and third dimensions, and of LUXD on the first dimension. In this way we created dimensions that correspond to partly overlapping clusters of variables.

The DEDICOM solution, subject to the above constraints, accounted for 85.3% of the total sum of squares, which is only 1.1% less than the unconstrained solution. The solution for A and R, with A normalized to unit sums of squares, is presented as the constrained overlapping solution in Table 1. The constrained overlapping solution revealed more or less the same pattern as the unconstrained DEDICOM solution. However, the present solution is simpler in that one does not have to account for small secondary or tertiary loadings for most of the car categories. To interpret this solution in a little more detail, note that the first dimension is related to plain large and midsize cars, and inversely related to domestic luxury cars. The second dimension represents big, luxury, and midsize specialty cars. The third dimension represents small/specialty cars and is inversely related to medium price standard cars. The numbers in R show that medium sized cars are often replaced by other medium sized cars, but also relatively often by small/specialty cars. Big cars are usually followed by big cars, and small/specialty cars are usually followed by small/specialty cars. So the most pronounced asymmetry is in the fact that medium sized cars tend to be replaced by small/specialty cars.

Variants For Off-Diagonal DEDICOM

Above, the standard DEDICOM procedure was used. This procedure can only be used when the elements on the diagonal are meaningful. In certain kinds of asymmetric tables, the diagonal is not meaningful, or not even defined at all. For instance, in friendship rating data it does not seem meaningful to rate one's friendship with oneself. In other cases, relations of an object to itself may be meaningful, but of a different nature than relations of an object to a different object. In both cases, the DEDICOM model can still be used, but the quality of the representation should be expressed in terms of the off-diagonal elements only (see Harshman, 1978). In the present section we will discuss variants of constrained DEDICOM that only fit the off-diagonal elements.

Takane (1985) suggested two algorithms for minimizing the sum of squared off-diagonal differences between X and ARA'. One of these, in line with a suggestion by Harshman et al. (1982, p. 209), adds an additional cycle to the algorithm to estimate the off-diagonal elements. That is, the algorithm proceeds by alternately updating A and R, and substituting the current diagonal elements of ARA' in X, until convergence. This procedure could be very slow. Ten Berge and Kiers (1989) provided a more efficient and monotonically convergent algorithm.

The purpose of the present section is to provide algorithms for handling the subspace constraints described above. A straightforward way to do so would be to use the constrained DEDICOM algorithms described above, supplemented with a diagonal estimation step. However, such a procedure is expected to be slow. In the rather common case where certain elements are fixed to zero, it is possible to adjust the ten Berge and Kiers (1989) algorithm so that it minimizes the off-diagonal DEDICOM function over A and R subject to that particular constraint on A. For all other constraints we can resort to the constrained DEDICOM algorithms proposed above, supplemented with a diagonal estimation step.

The off-diagonal DEDICOM algorithm proposed by ten Berge and Kiers (1989) minimizes the sum of squared off-diagonal differences by alternately updating R given A, and each row of A, given the other rows of A, and R. The updating of $\mathbf{a}_i'$ (the i-th row of A) consists of minimizing

$$g(\mathbf{a}_i) = \left\|\begin{pmatrix}\mathbf{c}_i\\ \mathbf{r}_i\end{pmatrix} - \begin{pmatrix}A_iR\\ A_iR'\end{pmatrix}\mathbf{a}_i\right\|^2, \tag{5}$$

over $\mathbf{a}_i$, where $\mathbf{c}_i$ and $\mathbf{r}_i$ denote the i-th column and row, respectively, of X, with the i-th elements deleted, and A_i denotes A with the i-th column deleted. Now suppose A is constrained to have zeros in certain prespecified places. Then updating $\mathbf{a}_i$, $i = 1, \ldots, n$, can be handled by deleting the zero elements in $\mathbf{a}_i$ (forming $\tilde{\mathbf{a}}_i$) and deleting the corresponding columns of $\binom{A_iR}{A_iR'}$, forming $\tilde{Z}$. Then it remains to minimize $\|\binom{\mathbf{c}_i}{\mathbf{r}_i} - \tilde{Z}\tilde{\mathbf{a}}_i\|^2$ over $\tilde{\mathbf{a}}_i$, which is a straightforward regression problem. To obtain the optimal $\mathbf{a}_i$ from $\tilde{\mathbf{a}}_i$, we substitute the elements of $\tilde{\mathbf{a}}_i$ at the positions for the nonzero elements of $\mathbf{a}_i$. Clearly, each of the rows of A can be updated in this way, for given R. Alternatively, for given A, R is updated as by ten Berge and Kiers (1989, p. 334, also see Appendix A, Equation (9)), where for the diagonal of X we substitute the diagonal of the current values of ARA'. Alternately updating the rows of A, and R in this way, we monotonically decrease the off-diagonal sum of squares of $(X - ARA')$. Because this function value is bounded below by zero, the procedure must converge to a stable function value.

Skew-symmetric DEDICOM With Different Constraints on Different "Bimensions"

Another interesting special case of DEDICOM is when it is applied to a skew-symmetric matrix. A skew-symmetric matrix is a matrix in which $x_{ij} = -x_{ji}$, $i, j = 1, \ldots, n$. Skew-symmetric matrices emerge most notably in cases where an asymmetric table is split into a symmetric part and a skew-symmetric part, both of which are analyzed separately. They can also occur in a more natural way, like in preference data, or debts/credits balance data. The skew-symmetric part is sometimes represented by a truncated singular value decomposition (SVD) that has a particularly simple form (Gower, 1977; also, see Constantine & Gower, 1978). That is, the singular values of a skew-symmetric matrix come in pairs of equal singular values (supplemented by one zero singular value if the order of the matrix is odd), and, if the left-hand singular vectors associated with such a pair of singular values are $\mathbf{p}_1$ and $\mathbf{q}_1$, then the right-hand singular vectors are $\mathbf{q}_1$ and $-\mathbf{p}_1$. Hence, if one writes U for the matrix of left-hand singular vectors, X can be written as $X = U\Sigma U'$, where Σ is the block-diagonal matrix with blocks $\begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}$, and a final 1×1 block with element 0 if the order is odd. This shows that the truncated SVD of a skew-symmetric matrix gives the DEDICOM representation for this matrix as well. Gower (1977), Harshman (1981b), Dawson and Harshman (1986), and Harshman and Lundy (1990) presented examples of the analysis of a skew-symmetric matrix. Interpretation of the solution is done "bimensionwise" (Harshman, 1981b), that is, by a pair of corresponding dimensions at a time. Specifically, the contribution of one bimension to the representation of x_{ij} equals $\pm$ the singular value times twice the area of the triangle formed by the origin and the points i and j in the bimension plane (with the sign depending on whether going from i to j along one of the sides of the triangle is clockwise $(-)$ or counterclockwise $(+)$).

Harshman (1981b) has suggested simple structure rotations of those bimensions to facilitate interpretation. If an arbitrary simple structure rotation is applied to A, this will typically cause the bimensional structure, giving separable contributions of different pairs of dimensions, to disappear. What does preserve bimensional structure (and maintains separability of contributions of different pairs of dimensions) is a rotation of the dimensions of one bimension by the same rotation matrix. To see this, one can rewrite the DEDICOM model for the skew-symmetric case as

$$\hat{X} = P\Delta Q' - Q\Delta P' = \Sigma_l \delta_l(\mathbf{p}_l\mathbf{q}_l' - \mathbf{q}_l\mathbf{p}_l'), \tag{6}$$

where Δ is a diagonal matrix with elements δ_l denoting the singular value corresponding to the l-th bimension, and $\mathbf{p}_l$ and $\mathbf{q}_l$ are the columns in A that correspond to the l-th bimension. Now, if one writes $\tilde{P} = P\Delta^{1/2}$, and $\tilde{Q} = Q\Delta^{1/2}$, then

$$\hat{X} = \tilde{P}\tilde{Q}' - \tilde{Q}\tilde{P}' = \tilde{P}TT'\tilde{Q}' - \tilde{Q}TT'\tilde{P}' = \Sigma_l \delta_l^*(\mathbf{u}_l\mathbf{v}_l' - \mathbf{v}_l\mathbf{u}_l'), \tag{7}$$

where T is an orthonormal matrix, $\mathbf{u}_l = \tilde{P}\mathbf{t}_l(\mathbf{t}_l'\tilde{P}'\tilde{P}\mathbf{t}_l)^{-1/2}$ and $\mathbf{v}_l = \tilde{Q}\mathbf{t}_l(\mathbf{t}_l'\tilde{Q}'\tilde{Q}\mathbf{t}_l)^{-1/2}$ are the unit normalized dimensions of the l-th rotated bimension, and δ_l^* denotes the "contribution" of this rotated bimension. Hence, orthonormal rotations of $\tilde{P}$ and $\tilde{Q}$ do not change $\hat{X}$, and, as is apparent from (7), yield again a bimensional structure, with separable contributions of different bimensions. These are not the only transformations that retain bimensional structure. For instance, rotation of the two dimensions of a bimension retains bimensional structure as well, and so does any combination of rotations within bimensions and the above (more complicated) type of rotations across bimensions. Harshman (personal communication, September 18, 1991; also, see Harshman, 1981b, p. 33) has proposed several procedures for using such combined rotations to obtain simple structure. However, nothing in these procedures seems to preclude that a (simply structured) solution is found in which the two dimensions of a bimension are related to different clusters of stimuli. For instance, the first dimension may be related to Stimuli 1, 2, 3, and 4, (and not to 5 through 8), while the second is related to Stimuli 1, 3, 5, and 7 (and not to 2, 4, 6, and 8). Such solutions do not adequately facilitate the interpretation of a bimension. It is preferred to have two dimensions of a bimension being related to the *same* cluster of stimuli (e.g., the whole bimension being related mainly to Stimuli 1, 2, 3, and 4). Instead of solving this problem of simple structure *rotation* of bimensions, we propose here, as in the previous sections, to *constrain* the solution to have a particular simple structure. Specifically, it is proposed that the same constraint, usually of simple structure, is imposed on both dimensions of a bimension, that is, on $\tilde{P}$ and $\tilde{Q}$.

To formulate the above proposal in terms of the original DEDICOM model, we rewrite the DEDICOM model for the skew-symmetric case as

$$\hat{X} = \tilde{P}\tilde{Q}' - \tilde{Q}\tilde{P}' = AR_IA', \tag{8}$$

where R_I is a fixed block-diagonal matrix with 2×2 blocks $J \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ along the diagonal, and $A = (\tilde{\mathbf{p}}_1 \vdots \tilde{\mathbf{q}}_1 \vdots \ \ldots \ \vdots \tilde{\mathbf{p}}_r \vdots \tilde{\mathbf{q}}_r)$. Thus, each pair of two consecutive columns of A contains the coordinates on one bimension. Note that we assume from now on that for skew-symmetric data A has an even number of dimensions, because otherwise a bimensional representation is not possible. Moreover, if the dimensionality is not even, the last dimension does not contribute to the fit (Harshman, 1981b). To impose *bimen-*

sionwise simple structure constraints on $\tilde{P}$ and $\tilde{Q}$ implies that, if certain elements of one column of A are fixed to zero, then so are the corresponding elements of the other column that belong to the same bimension.

We distinguish two possible cases, one in which a simple structure with nonoverlapping clusters is imposed, and the other in which simple structure with overlapping clusters is imposed. The first case is simple to solve, because the solution can be found from separate DEDICOM analyses. For example, suppose we have a 10×10 skew-symmetric data matrix, and in the first two columns of A the first four elements are fixed to zero, and in the next two columns of A, the remaining six elements are fixed to zero. Then, one can write $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, with A_1 of order 4×2 and A_2 of order 6×2. Hence, the model can be written as

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = AR_IA' + E = \begin{pmatrix} A_1JA_1' + E_{11} & E_{12} \\ E_{21} & A_2JA_2' + E_{22} \end{pmatrix},$$

where $E_{11}, \ldots, E_{22}$ denote submatrices of the error matrix E. To fit this model reduces to the two separate DEDICOM problems of fitting X_{11} and X_{22}.

If the simple structure constraints imply overlapping clusters, one can no longer turn to fitting different DEDICOM models separately. To fit a model with constraints in terms of overlapping clusters we use the observation that, for DEDICOM on skew-symmetric data, one can use both the columnwise procedure for fixed diagonal DEDICOM, and the rowwise procedure for off-diagonal DEDICOM. The latter procedure will always maintain the diagonal elements of ARA' equal to zero, as follows from the fact that R will be skew-symmetric. Hence, diagonal value estimates always produce zeros on the diagonal of X. Therefore, to fit the DEDICOM model with zero-constraints to a skew-symmetric matrix, one can use the adjusted rowwise ten Berge and Kiers (1989) algorithm described in the previous section, with R kept fixed to R_I. In this way, one can deal with bimensionwise constraints for producing overlapping clusters of stimuli, by using an algorithm already described above.

As an example we reanalyzed a data set given by Wiepkema (1961; see also van der Heijden, 1987, pp. 126–130) on frequencies of 12 different courtship behaviors (rows) being followed by any of these courtship behaviors (columns) of 13 bitterlings. The 12 different behaviors are jerking (JK), turning beats (TB), head butting (HB), chasing (CHS), fleeing (FL), quivering (QU), leading (LE), head down posture (HDP), skimming (SK), snapping (SN), chafing (CHF), and fin flickering (FFL). We analyzed the skew-symmetric part of this data set, and found that the one- and two-bimensional solutions accounted for 94.8% and 97.5%, respectively, of the total sum of squares. Note that the skew-symmetric part describes the frequency of nonreciprocal behavior sequences. In Table 2 the (unrotated) two-bimensional solution is reported. It should be noted that these results are based on the principal axes, which are, within bimensions, rotationally undetermined. On the basis of these results, we chose to constrain the bimensions such that the first is associated only with the behaviors QU, LE, HDP, and SK (the sexual behaviors), and the second with the other (aggressive and non-reproductive behaviors). This constrained DEDICOM problem reduces to two separate skew-symmetric DEDICOM analyses, which together turned out to account for 94.3% of the total sum of squares.

Obviously, the above constrained solution is simpler than the unconstrained two-bimensional solution, but the fit is also markedly poorer, and does not even exceed that of the one-bimensional solution. Therefore, we decided to constrain the bimensions such that they correspond to partly overlapping clusters of behavors. The first bimension again pertained to the sexual behaviors QU, LE, HDP, and SK, and the second

TABLE 2

(Constrained) DEDICOM Solution of the Skew-Symmetric Part of the Wiepkema Data

	unconstrained solution				constrained solution			
	bim.1		bim.2		bim.1		bim.2	
matrix A								
JK	-.05	.00	-.27	-.68			.01	-.26
TB	.00	-.01	-.60	.20			.18	-.09
HB	-.03	.02	.11	.47			-.14	.33
CHS	-.02	-.10	-.32	.13			.51	.08
FL	.03	.00	-.13	.21			.06	.13
QU	-.19	-.55	.48	-.04	-.66	.12	-.73	.10
LE	-.42	-.02	-.01	-.38	-.20	-.35		
HDP	-.15	.82	.26	-.01	.63	-.45		
SK	.87	.00	.12	-.22	.37	.81	-.11	-.89
SN	-.06	-.09	-.14	.15			.37	-.03
CHF	.02	-.02	.06	-.02			.03	-.05
FFL	-.02	.00	.30	.04			-.07	-.01
matrix R								
	0.0	-104.8	0.0	0.0	0.0	112.5	0.0	0.0
	104.8	0.0	0.0	0.0	-112.5	0.0	0.0	0.0
	0.0	0.0	0.0	-17.2	0.0	0.0	0.0	26.1
	0.0	0.0	17.2	0.0	0.0	0.0	-26.1	0.0
fit	97.5 %				95.8 %			

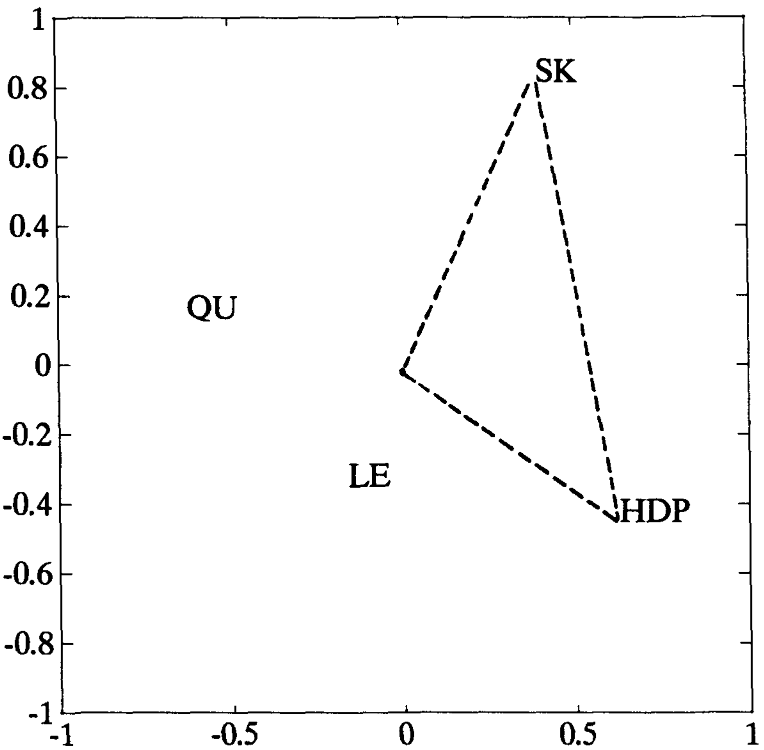

FIGURE 1.
First constrained bimension of the Wiepkema data.

was now associated with TB, HB, CHS, FL, QU, SK, SN, CHF, and FFL. Note that these bimensions overlap in terms of the behaviors QU and SK. The solution thus constrained accounted for 95.8%. To interpret this solution, one can focus on the relations between behaviors within bimensions, as plotted in Figures 1 and 2. The first bimension gives by far the largest elements in R (see Table 2), so apparently most of the skew-symmetric relations is found between sexual behaviors. To interpret these skew-symmetric relations one may compute the areas of certain triangles (see Figure 1) to see to what extent a particular behavior is followed *more often by the other than vice versa.* For example, the head down posture (HDP) is followed much more often by skimming (SK) than the other way around (triangle has a large area). In fact, it is easy to see that particularly large triangles are found while going from quivering to leading behavior, from leading to head down posture, from head down posture to skimming, and from skimming to quivering, which may therefore well establish a dominant circular order of sexual behaviors of bitterlings. Since this bimension accounts for most of the skew-symmetry in the data, it seems that this sequence of behaviors can well be seen as the most dominant sequence of nonreciprocal behaviors. The second bimension can be interpreted analogously, but it should be taken into account that this bimension is not very strong (as indicated by the associated elements in R).

Discussion

Above, a series of methods have been discussed for DEDICOM analysis subject to various constraints. The algorithms to fit DEDICOM subject to these constraints have been given, and the methods have been illustrated by means of example analyses. In these analyses we imposed constraints that were mainly chosen on the basis of the

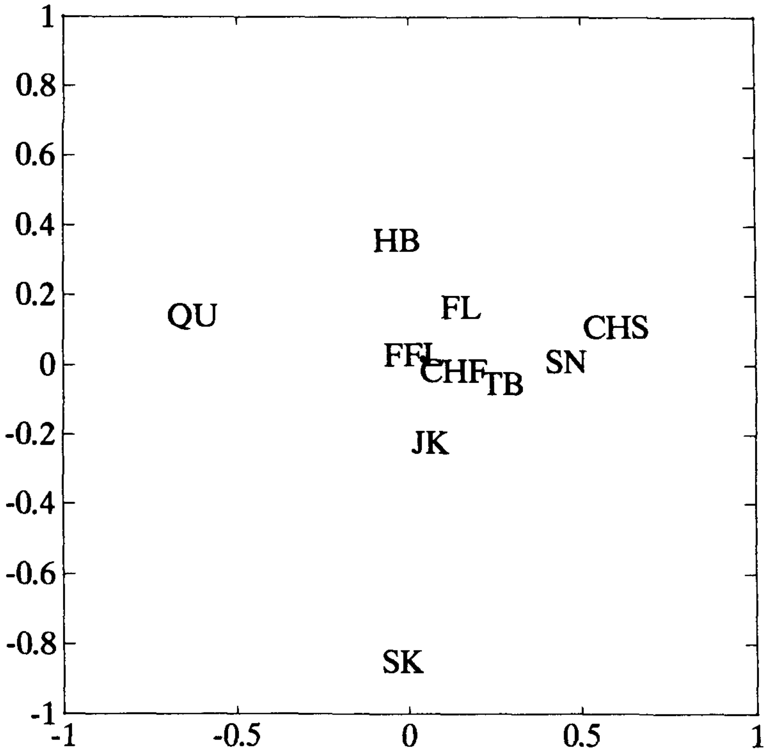

FIGURE 2.
Second constrained bimension of the Wiepkema data.

unconstrained analyses. Alternatively, one may use prior knowledge on the stimuli (if available), and assess whether the imposed structure explains most of the information. Combinations of these strategies will probably be most fruitful.

The idea of using different constraints on different dimensions has been proposed by Takane et al. (1991) for constraining dimensions in principal components analysis (PCA). A prevalent alternative for PCA is common factor analysis, where a covariance matrix is approximated by the product moment of a loading matrix, and the diagonal elements of the covariance matrix (communalities) are estimated as well. A well-known procedure for this is MINRES (Harman & Jones, 1966). Since MINRES can be seen as applying off-diagonal DEDICOM to the covariance matrix (ten Berge & Kiers, 1989), the present procedure for dimensionwise constrained DEDICOM can also be used as a method for dimensionwise constrained MINRES factor analysis.

Another extension of the constrained DEDICOM methods described above is found by imposing constraints on the three-way variant of DEDICOM proposed by Harshman (1978; see also Harshman et al., 1982). If $X_1, \ldots, X_p$ denote p asymmetric data matrices of the same order, the three-way DEDICOM model can be written as

$$X_k = AD_kRD_kA' + E_k,$$

where A and R have the same meaning as in two-way DEDICOM, $D_1, \ldots, D_p$ are diagonal matrices denoting the differential importance of the "basic aspects" in the p different data sets, and $E_1, \ldots, E_p$ denote matrices with error terms. An algorithm to fit this model has recently been proposed by Kiers (in press). His algorithm can easily be adjusted to allow for general subspace constraints, or different constraints on different dimensions of A.

In the present paper a number of different iterative algorithms have been proposed. No mention has been made of how the parameters should be initialized. In the analyses carried out here, A was initialized randomly. Because the procedure usually converged fast, there seems to be little reason to look for better starts. In case one does want to use rational starts, we suggest to use the first eigenvector of $G_l G_l'(X + X')G_l G_l'$ as a start for the l-th column of A, $l = 1, \ldots, r$, in the case of different constraints on different dimensions. In the case of different constraints on different bimensions, one might use a varimax rotation of the bimensions as an initial configuration for A, with the elements constrained to be zero set equal to zero.

In some of the analyses we imposed the constraint that objects should "belong" to nonoverlapping clusters of variables. However, as remarked by an anonymous reviewer, some caution should be taken in imposing such constraints in DEDICOM. That is, in case objects are associated with only one dimension (cluster), the relations between such objects assigned to the same cluster are modeled to be symmetric by definition, which need not be the explicit intention of the user. Indeed, it can be judged reasonable to model the relationships between objects belonging to the same cluster in a symmetric way, since their membership in the same cluster indicates that they "behave" similarly. However, if it is desired to model relationships within clusters asymmetrically as well, then the DEDICOM model with nonoverlapping clusters is too restricted for that purpose, and one might resort to more general models, for instance, the dual domain DEDICOM model (see Harshman et al., 1982, p. 239), in which $X = ARB' + E$, with A and B giving weights for the relations of the row-objects to idealized row-objects, and for the column-objects to idealized column-objects, respectively. In this approach, using nonoverlapping clusters does not entail certain forced symmetric modelings. However, the transition from the single domain DEDICOM model to the dual DEDICOM model is accompanied by a considerable loss in parsimony. It should be emphasized once more that the above phenomenon only occurs for objects that are associated with one single dimension. As a result, in skew-symmetric DEDICOM, using nonoverlapping clusters does not entail such forced symmetric modelings, because each object is associated with one *bi*mension, and hence with more than one single dimension.

Appendix
An Alternating Least Squares Algorithm for DEDICOM Subject to Different Constraints on Different Dimensions

To minimize the DEDICOM loss function (2) subject to the constraint that $\mathbf{a}_l = G_l \mathbf{u}_l$, one can use an alternating least squares algorithm as follows. The minimum of σ over R for given A is attained for

$$R = (A'A)^{-1}A'XA(A'A)^{-1}; \tag{9}$$

see Penrose (1956). If $A'A$ is singular, the inverse should be replaced by a generalized inverse. The minimum of σ over A for given R is more difficult to obtain. We propose to minimize σ over A columnwise. That is, σ can be expressed as a function of the l-th column of A as

$$\sigma(\mathbf{a}_l) = \|X - \sum_{j=1}^{r} \sum_{k=1}^{r} r_{jk}\mathbf{a}_j\mathbf{a}_k'\|^2 = \left\| \left(X - \sum_{j \neq l} \sum_{k \neq l} r_{jk}\mathbf{a}_j\mathbf{a}_k'\right) - \left(\sum_{j \neq l} r_{jl}\mathbf{a}_j\right)\mathbf{a}_l' - \mathbf{a}_l\left(\sum_{j \neq l} r_{lj}\mathbf{a}_j'\right) - r_{ll}\mathbf{a}_l\mathbf{a}_l' \right\|^2. \tag{10}$$

The constraint Diag $(A'A) = I_r$ implies $\mathbf{a}_l'\mathbf{a}_l = 1$, and hence $\mathbf{u}_l'\mathbf{u}_l = 1$. If one defines $X_{-l} \equiv (X - \sum_{j\neq l} \sum_{k\neq l} r_{jk}\mathbf{a}_j\mathbf{a}_k')$, $\mathbf{y}_l \equiv (\sum_{j\neq l} r_{jl}\mathbf{a}_j)$, and $\mathbf{z}_l \equiv (\sum_{j\neq l} r_{lj}\mathbf{a}_j)$, and substitutes $\mathbf{a}_l = G_l\mathbf{u}_l$ for $\mathbf{a}_l$, (10) can be rewritten as

$$\begin{aligned}
\sigma(\mathbf{u}_l) &= \|X_{-l} - \mathbf{y}_l\mathbf{u}_l'G_l' - G_l\mathbf{u}_l\mathbf{z}_l' - r_{ll}G_l\mathbf{u}_l\mathbf{u}_l'G_l'\|^2 \\
&= c_l - 2 \text{ tr } X_{-l}'\mathbf{y}_l\mathbf{u}_l'G_l' - 2 \text{ tr } X_{-l}'G_l\mathbf{u}_l\mathbf{z}_l' - 2r_{ll} \text{ tr } X_{-l}'G_l\mathbf{u}_l\mathbf{u}_l'G_l' \\
&\quad + 2 \text{ tr } G_l\mathbf{u}_l\mathbf{y}_l'G_l\mathbf{u}_l\mathbf{z}_l' + 2r_{ll} \text{ tr } G_l\mathbf{u}_l\mathbf{y}_l'G_l\mathbf{u}_l\mathbf{u}_l'G_l' + 2r_{ll} \text{ tr } \mathbf{z}_l\mathbf{u}_l'G_l'G_l\mathbf{u}_l\mathbf{u}_l'G_l' \\
&= c_l + (-2\mathbf{y}_l'X_{-l}G_l - 2\mathbf{z}_l'X_{-l}'G_l + 2r_{ll}\mathbf{y}_l'G_l + 2r_{ll}\mathbf{z}_l'G_l)\mathbf{u}_l \\
&\qquad + \mathbf{u}_l'(2G_l'\mathbf{y}_l\mathbf{z}_l'G_l - 2r_{ll}G_l'X_{-l}'G_l)\mathbf{u}_l, \qquad (11)
\end{aligned}$$

where c_l is a constant not depending on $\mathbf{u}_l$. To minimize (11) we use a procedure similar to the one proposed by Kiers (1989). That is, we first replace the asymmetric matrix $(2G_l'\mathbf{y}_l\mathbf{z}_l'G_l - 2r_{ll}G_l'X_{-l}'G_l)$ by its symmetric part $S \equiv (G_l'(\mathbf{y}_l\mathbf{z}_l' + \mathbf{z}_l\mathbf{y}_l')G_l - r_{ll}G_l'(X_{-l} + X_{-l}')G_l)$, and compute the eigendecomposition

$$S = K\Lambda K', \qquad (12)$$

with Λ diagonal, and K orthonormal. Next, we define

$$\mathbf{w}_l \equiv K'G_l'(X_{-l}'\mathbf{y}_l + X_{-l}\mathbf{z}_l - r_{ll}\mathbf{y}_l - r_{ll}\mathbf{z}_l), \qquad (13)$$

and

$$\mathbf{b}_l \equiv K'\mathbf{u}_l, \qquad (14)$$

so that (11) can be written as

$$\sigma(\mathbf{b}_l) = c_l - 2\mathbf{w}_l'\mathbf{b}_l + \mathbf{b}_l'\Lambda\mathbf{b}_l. \qquad (15)$$

The problem of minimizing this function subject to the constraint that $\mathbf{b}_l'\mathbf{b}_l = \mathbf{u}_l'KK'\mathbf{u}_l = \mathbf{u}_l'\mathbf{u}_l = 1$, has been solved by ten Berge and Nevels (1977). Having found the optimal $\mathbf{b}_l$ we can obtain the optimal $\mathbf{u}_l$ from $\mathbf{u}_l = K\mathbf{b}_l$ and, finally, obtain the optimal $\mathbf{a}_l$ as $\mathbf{a}_l = G_l\mathbf{u}_l = G_lK\mathbf{b}_l$. In this way we can update each column of A successively. Each updating of a column of A decreases (or at least does not increase) the loss function, so alternately updating the columns of A by the procedure described above, and R according to (9), monotonically decreases the loss function. Because the loss function is bounded below by zero, this procedure will converge to a stable function value.

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