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Choice model analysis of the "pick any/*n*" type of binary data[1]

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Abstract: In the "pick any/*n*" method, subjects are asked to choose any number of items from a list of *n* items according to some criterion. This kind of data can be analyzed as a special case of either multiple-choice data or successive categories data where the number of response categories is limited to two. An item response model was proposed for the latter case, which is a combination of an unfolding model and a choice model. The marginal maximum-likelihood estimation method was developed for parameter estimation to avoid incidental parameters, and an expectation-maximization algorithm used for numerical optimization. Two examples of analysis are given to illustrate the proposed method, which we call MAXSC.

Key words: multiple-choice data, successive categories data, unfolding model, marginal maximum-likelihood (MML) method, expectation-maximization algorithm.

In the "pick any/*n*" method, subjects are given a set of *n* items (stimuli, objects) and are asked to choose any number of them according to some criterion. For example, they may be asked to choose acceptable vacation sites from a list of *n* possible sites; they may be asked to choose from a list of *n* durable goods those they intend to purchase in the next six months; in a personality inventory they may be asked to mark those behaviors that adequately describe their behavioral disposition.

These data may be analyzed as a special case of multiple-choice (unordered categorical) data with only two response categories ("acceptable" and "not acceptable"; "purchase" and "do not purchase"; "apply" and "does not apply") per item. An item response model for such data has been presented (Takane, 1997). In this model both of the responses to each item are represented as points in a multidimensional space, and subjects choose one according to its closeness to their ideals (represented as points in the same space). Alternatively, the data may be treated as a special type of successive categories (ordered categorical) data with only two response categories. In this case each item, rather than each category of item, may be represented as a point, and the subjects are assumed to choose (or not to choose) the item according to its closeness to their ideals. In this paper we discuss a model for "pick any/*n*" data viewed as a special type of successive categories data.

Subjects' ideal points account for individual differences in the kind of preference data we deal with in this paper. However, they also become incidental parameters (i.e., parameters which are bound to increase in number as more observations are obtained), if estimated jointly with other parameters of interest. This

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may result in inconsistent maximum-likelihood (ML) estimates (e.g., Andersen, 1980) of non-incidental parameters. To avoid this difficulty, the marginal maximum-likelihood (MML) estimation method (Bock & Lieberman, 1970) has been developed for the proposed model. In this estimation method, a subject's ideal point is introduced as a random effect parameter, assumed to follow a certain distribution over the population of subjects, which is then integrated out to obtain marginal probabilities of choice patterns. Estimates of item parameters as well as those characterizing the subject distribution are determined so as to maximize the marginal likelihood.

In the next section we present details of the proposed model along with the MML method for parameter estimation. We call this method MAXSC, MAXimum likelihood item response theory (IRT) models for Successive Categories data. We then discuss an expectation maximization (EM) algorithm for maximizing the marginal likelihood. This is followed by examples of the model's application, and finally by a discussion and a look at further prospects.

The model

Suppose each of N subjects has chosen any number of items from a set of n items according to a certain criterion. It is convenient to classify the N subjects in terms of their choice patterns, which are indexed by k. Let us assume that there are K distinct observed choice patterns. Define:

$$g^*_{ki} = \begin{cases} 1, \text{ if item } i \text{ is chosen in} \\ \text{response pattern } k \\ 0, \text{ otherwise.} \end{cases} \quad (1)$$

Each choice pattern is then described by a sequence of g^*_{ki} for $i = 1, \ldots, n$. Let f_k denote the observed frequency of the kth choice pattern. We assume that items are represented as points in an A-dimensional Euclidean space. Let x_{ia} denote the coordinate of item i on dimension a ($i = 1, \ldots, n$; $a = 1, \ldots, A$). We also assume that the subject's ideal point is represented in the same Euclidean space. Let $\mathbf{y} = (y_1, \ldots, y_A)'$ be a vector of coordinates of the subject point. We further assume that $\mathbf{y}$ is a random vector with its density function denoted by $h(\mathbf{y})$. Let $d_i(\mathbf{y})$ denote the distance between item i and a subject whose ideal point is at $\mathbf{y}$. This is given by:

$$d_i(\mathbf{y}) = \{\sum_a^A (x_{ia} - y_a)^2\}^{1/2}. \quad (2)$$

This is a special case of Coombs' (1964) unfolding (ideal point) model for preference data. Let $p_i(\mathbf{y})$ denote the conditional probability of a subject at $\mathbf{y}$ choosing item i. We posit that:

$$p_i(\mathbf{y}) = \frac{\exp(-d_i^v(\mathbf{y}))}{\exp(-d_i^v(\mathbf{y})) + b}, \quad (3)$$

where b is a threshold parameter. The model postulates that each item has a response strength, $\exp(-d_i^v(\mathbf{y}))$, for a subject at $\mathbf{y}$ which is a decreasing function of the distance between item i and $\mathbf{y}$, and that an item is chosen (or not chosen) by the subject according to its response strength relative to the response threshold. This is a special case of Luce's (1959) choice model, in which item i's response strength is compared with the response threshold, b, and the choice is made with the probability proportional to item i's response strength. That is, $p_i(\mathbf{y}) = c \exp(-d_i^v(\mathbf{y}))$ for some $c \neq 0$, but since the probabilities of choice and non-choice should add up to unity, c must be equal to $1/\{\exp(-d_i^v(\mathbf{y})) + b\}$. It is assumed that there is a single threshold parameter, b, that applies to all items. This means that the acceptance criterion is the same across all items. The constant, v, in the exponent modulates the shape of the response strength function. It can be set to either 1.0 or 2.0. However, for reasons discussed by Takane (1997), $v = 2.0$ is generally favored, and it will be assumed so throughout the rest of this paper.

The conditional probability, $P_k(\mathbf{y})$, of choice pattern k is now stated as:

$$P_k(\mathbf{y}) = \prod_i p_i(\mathbf{y})^{g^*_{ki}}(1 - p_i(\mathbf{y}))^{(1-g^*_{ki})}. \quad (4)$$

This assumes independence of choices conditional on $\mathbf{y}$, which is known as the local independence assumption in latent structure analysis (Lazarsfeld & Henry, 1968). The marginal probability, P_k, of choice pattern k is then given by

$$P_k = \int P_k(\mathbf{y})h(\mathbf{y})d\mathbf{y}. \tag{5}$$

We assume

$$\mathbf{y} \sim N(\mathbf{0}, \Sigma), \tag{6}$$

where Σ is further assumed to be diagonal without loss of generality, with the diagonal elements denoted by σ_a^2, $a = 1, \ldots, A$. The zero mean vector in (6) identifies the origin of the representation space. The integral in (5) may be approximated by the Gaussian quadrature

$$P_k = \sum_q P_k(\mathbf{y}_q)B(\mathbf{y}_q) \tag{7}$$

at selected points $\mathbf{y}_q$. A special table is available for $B(y_{qa})$ with $B(\mathbf{y}_q) = \Pi_a B(y_{qa})$. The likelihood of the entire set of choice patterns can now be stated as:

$$L = \prod_k^K P_k^{f_k}. \tag{8}$$

We find estimates of model parameters, x_{ia}, b, and σ_a^2 for $i = 1, \ldots, n$ and $a = 1, \ldots, A$ that maximize the L.

Once L in (8) is maximized, we may use the Akaike information criterion (AIC; Akaike, 1974) defined by:

$$AIC(\pi) = -2 \ln L^*(\pi) + 2n_\pi, \tag{9}$$

for the goodness of fit (GOF) comparison among fitted models, where π represents a particular model fitted, $L^*(\pi)$ is the maximum likelihood obtained under model π, and n_π is the effective number of parameters in the model. The model with the smallest value of AIC is chosen as the best predictive model of future observations. The effective number of parameters is calculated by:

$$n_\pi = (n + 1)A + 1. \tag{10}$$

There are nA x_{ia} values, one b, and A values of σ_a. Note that the translational and rotational indeterminacies in the Euclidean space have been removed by setting the mean vector of $\mathbf{y}$ to zero and the covariance matrix to be diagonal in (6).

Some IRT models similar to the one presented in this paper have been proposed by various authors, including DeSarbo and Hoffman (1986, 1987), Andrich (1988), and Hoijtink (1990). These models, as in the present case, all use distance functions (the unfolding model), as opposed to scalar products, in the exponent of the response strength function. DeSarbo and Hoffman, however, used the joint estimation method, in which subjects' ideal points were simultaneously estimated along with other parameters of interest. The potential difficulty associated with the lack of usual asymptotic properties of ML estimators under such circumstances has already been noted. Andrich's model has a serious limitation in that it cannot predict choice probabilities greater than .5, as Andrich himself noted. This is due to the unnecessarily restrictive assumption that $b = 1$ in Andrich's model. Hoijtink's method, called Parella, allows a more flexible response strength function than (3) adopted in this paper. However, both Parella and Andrich's simple logist model are limited to a single dimension.

An EM algorithm

Following Bock and Aitkin (1981), we have developed an EM algorithm to maximize the log of the marginal likelihood stated in (8). The algorithm comprises the following two steps, which are alternated until convergence is reached.

1. E-step. For fixed parameter estimates, calculate:

$$\bar{g}_{qi} = B(\mathbf{y}_q)\sum_k g_{ki}P_k(\mathbf{y}_q)/P_k, \tag{11}$$

where $g_{ki} = f_k g^*_{ki}$, and

$$\bar{f}_q = B(\mathbf{y}_q)\sum_k f_k P_k(\mathbf{y}_q)/P_k. \tag{12}$$

2. M-step. For each i independently, maximize

$$l_i = \sum_q \{\bar{g}_{qi} \ln p_i(\mathbf{y}_q) + (\bar{f}_q - \bar{g}_{qi}) \ln (1 - p_i(\mathbf{y}_q))\} \tag{13}$$

with respect to x_{ia}, and after all x_{ia}s ($i = 1, \ldots, n$) are updated, maximize

$$l = \sum_i l_i \tag{14}$$

with respect to b and σ_a^2 ($a = 1, \ldots, A$). In the current algorithm these two sets of parameters are updated sequentially, although this is merely a convention adopted here. Maximizations in the M-step are carried out by Fisher's scoring algorithm. This algorithm is particularly attractive, since the number of parameters to be updated simultaneously is small (at most A). We use Hayashi's (1952) third kind of quantification method to obtain initial estimates of item parameters. (This has an inadvertent effect of forcing the choice pattern in which no items are chosen to be excluded from analysis by MAXSC, although the proposed model itself should be able to accommodate such a pattern.) We simply set $b = 3.00$ and $\sigma_a^2 = 1/A$ for all a, initially.

After the convergence is reached, the subject's ideal point can, *a posteriori*, be estimated for each choice pattern. We may use the Bayes' expected *a posteriori* (EAP) estimation method for this purpose (Bock & Aitkin, 1981). The distributional assumption on $\mathbf{y}$ in (6) can also be relaxed by re-estimating the quadrature weights, $B(\mathbf{y}_q)$, in (7) from empirical data in each iteration. The formulae for the EAP estimates of the subject's ideal points as well as their variance and covariance estimates, and those for re-estimating the quadrature weights, have been given in Takane (1997) for a similar situation (MAXMC, MAXimum likelihood item response models for Multiple-Choice data). They can be used in the present context without any serious modifications.

Examples

We illustrate MAXSC with two examples of application. The first data set is Andrich's (1988). It has been analyzed by Andrich himself using his own model. The second data set comes from Sugiyama (1975). It has been analyzed by Heiser (1981) using homogeneity analysis, also known as Hayashi's (1952) third kind of quantification method, dual scaling (Nishisato, 1980), and multiple correspondence analysis (Greenacre, 1984).

Andrich's capital punishment data

This example pertains to attitudes toward capital punishment. A scale was formed consisting of eight attitude statements using Thurstone's methods. These statements are listed in Appendix A in order of least favorable to most favorable to capital punishment according to Wohlwill (1963) (as well as Thurstone's original study quoted in Wohlwill). Fifty-four graduate students at Mudock University in Australia taking an introductory course in measurement and statistics responded to the eight items. The 54 subjects fell into 22 distinct choice patterns out of 256 possible patterns. Observed frequencies of the 22 choice patterns are given in Table 3. Table 1 gives a summary of the GOF statistics (AIC). All solutions with MAXSC were obtained with five quadrature points along each dimension. One-, two-, and three-dimensional solutions were obtained by MAXSC. The minimum AIC criterion indicates that the three-dimensional solution is best among the fitted models, including the saturated model. The appropriate dimensionality could be higher, because no attempts were made to obtain solutions in higher dimensionalities. However, caution should be exercised, and not too much faith should be put on the values of AIC, since the sample size is so small with this data set that the asymptotic properties of the maximum-likelihood estimates are not likely to hold. Rather, from the slight decreases in the value of AIC beyond the dimensionality of 1, we may argue that the eight-item scale

Table 1. Goodness of fit comparison: Andrich's (1988) data

Saturated model	407.3	(255)
Dimensionality		
1	51.6	(10)
2	36.4	(19)
3	31.9	(28)

AIC (number of para.).

Table 2. Parameter estimates: Andrich's (1988) data

Statement	Coordinate
1	−0.52
2	−0.51
3	−0.50
4	0.09
5	1.72
6	1.86
7	2.03
8	2.28

Threshold estimate = 2.82.
Variance estimate = 1.18.

Table 3. EAP estimates of coordinates of choice patterns: Andrich's (1988) data

Choice pattern	Observed frequency	Coordinate	Standard error
01100000	4	−1.26	.52
11100000	10	−1.12	.63
01110000	3	−.50	.70
11110000	8	−.31	.60
01111000	1	.00	.06
11111100	2	.00	.04
01110010	1	.01	.09
10111100	1	.06	.29
01101010	1	.48	.69
01111110	1	.98	.69
10011011	1	1.47	.09
00111101	2	1.47	.11
01011111	1	1.47	.02
01001110	2	1.47	.06
00101101	2	1.47	.04
01001101	1	1.47	.04
10001111	2	1.47	.02
00011111	3	1.47	.02
00010011	1	1.47	.08
00001111	5	1.57	.39
00001100	1	1.69	.56
00000111	1	1.95	.74

Total, 54.

is essentially unidimensional with a few outlying choice patterns (e.g., 10011011, 10001111), contributing to the emergence of spurious dimensions. (Another potential cause will be mentioned in the Discussion.)

The unidimensional solution is displayed in Table 2. These are the estimated scale values of the eight items. These estimates are in exactly the same order as those obtained in two previous studies, reported by Wohlwill (1963). There is one reversal of order in Andrich's (1988) estimates which are obtained from the same data set as used in the present study. However, the amount of reversal is slight, and so we may conclude that the agreement is quite good among the four studies. The EAP estimates of choice patterns are presented in Table 3, along with their standard errors. They are listed in order of the estimated coordinate values, which are again closely in line with Andrich's results. In the first column of the table, where choice patterns are indicated as sequences of g^*_{ki}, we observe that ones run from upper left to lower right, indicating approximate unidimensionality of the eight items (with a few exceptions noted earlier).

Sugiyama's religious behavior data

Sugiyama (1975) investigated religious practice among Japanese people. The six questions used in the study are given in Appendix B. The data were collected from a total of 4,243 subjects, of whom 718 chose none of the six items and were excluded from the analysis. The remaining 3,525 subjects were classified into 60 distinct choice patterns. Three choice patterns marked "%" in Table 4 were unobserved. Observed frequencies of choice patterns are shown in Table 4. As before, one- to three-dimensional solutions were obtained, all with five quadrature points along each dimension. Resulting GOF statistics (AIC) are presented in Table 5. The saturated model turned out to be the best-fitting model according to the minimum AIC criterion. In this case the sample size is so large that models with more parameters tend to be favored by the criterion. It is also possible that some important element is missing in the current MAXSC model (see Discussion).

Figure 1 depicts the derived two-dimensional configuration. This choice was dictated by the interpretability consideration. The horizontal

Table 4. Frequencies of choice patterns in Sugiyama's (1975) data

No.	Choice pattern	Frequency
1	1 1 1 1 1 1	42
2	1 1 1 1 1 0	33
3	1 1 1 1 0 1	6
4	1 1 1 1 0 0	17
5	1 1 1 0 1 1	12
6	1 1 1 0 1 0	29
7	1 1 1 0 0 1	8
8	1 1 1 0 0 0	82
9	1 1 0 1 1 1	51
10	1 1 0 1 1 0	69
11	1 1 0 1 0 1	20
12	1 1 0 1 0 0	54
13	1 1 0 0 1 1	34
14	1 1 0 0 1 0	124
15	1 1 0 0 0 1	27
16	1 1 0 0 0 0	317
17	1 0 1 1 1 1	1
18	1 0 1 1 1 0	2
19%	1 0 1 1 0 1	0
20	1 0 1 1 0 0	9
21	1 0 1 0 1 1	1
22	1 0 1 0 1 0	11
23	1 0 1 0 0 1	7
24	1 0 1 0 0 0	59
25	1 0 0 1 1 1	8
26	1 0 0 1 1 0	23
27	1 0 0 1 0 1	7
28	1 0 0 1 0 0	35
29	1 0 0 0 1 1	10
30	1 0 0 0 1 0	55
31	1 0 0 0 0 1	13
32	1 0 0 0 0 0	194
33	0 1 1 1 1 1	11
34	0 1 1 1 1 0	7
35	0 1 1 1 0 1	2
36	0 1 1 1 0 0	5
37	0 1 1 0 1 1	4
38	0 1 1 0 1 0	8
39	0 1 1 0 0 1	4
40	0 1 1 0 0 0	44
41	0 1 0 1 1 1	72
42	0 1 0 1 1 0	126
43	0 1 0 1 0 1	45
44	0 1 0 1 0 0	142
45	0 1 0 0 1 1	80
46	0 1 0 0 1 0	258

Table 4. Continued

No.	Choice pattern	Frequency
47	0 1 0 0 0 1	137
48	0 1 0 0 0 0	760
49%	0 0 1 1 1 1	0
50	0 0 1 1 1 0	2
51%	0 0 1 1 0 1	0
52	0 0 1 1 0 0	4
53	0 0 1 0 1 1	4
54	0 0 1 0 1 0	3
55	0 0 1 0 0 1	6
56	0 0 1 0 0 0	30
57	0 0 0 1 1 1	33
58	0 0 0 1 1 0	48
59	0 0 0 1 0 1	38
60	0 0 0 1 0 0	64
61	0 0 0 0 1 1	42
62	0 0 0 0 1 0	96
63	0 0 0 0 0 1	90
64#	0 0 0 0 0 0	718

%: zero frequency patterns.
#: not included in the MAXSC analysis.

Table 5. Goodness-of-fit comparison: Sugiyama's (1975) data

Saturated model	39.7	(62)
Dimensionality		
1	974.2	(8)
2	530.0	(15)
3	472.9	(22)

AIC (number of para.).

direction seems to represent a contrast between authentic religious conduct (items 1 and 3) versus superstitious behavior (items 4, 5 and 6). On the vertical axis item 2 stands alone at the top, while all the other items are located at the bottom. This could be due to ambiguity in item 2. Many people visit their family graves once or twice a year as a routine family event, but it may have little to do with their religious beliefs. The population of the subjects has a much larger spread along the horizontal direction than the vertical. The variance estimate was .510 along the horizontal as opposed to

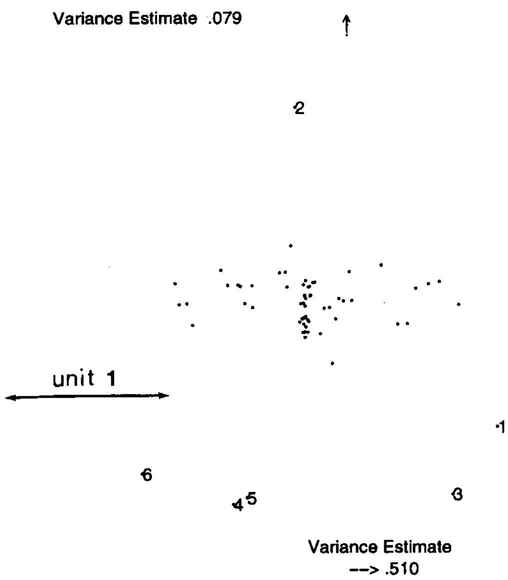

Figure 1. The two-dimensional configuration for Sugiyama's (1975) data.

.079 along the vertical. The estimate of the threshold parameter turned out to be 4.02.

Discussion and further prospects

We have presented an item response model for the "pick any/*n*" type of binary data, which combined Coombs' unfolding (ideal point) model and Luce's choice model. The MML estimation method and an associated EM algorithm have been developed to fit the model. The general approach presented in this paper points to an important direction in modeling multivariate categorical data (Takane, 1997) in general. The approach has already been extended to another type of categorical data by Hojo (1997).

We have seen two examples of the application of MAXSC. The performance of the model has turned out to be somewhat disappointing, however. In neither example could we adopt the minimum AIC solution as the best solution. Although part of the problem is the sample size, we also see some potential limitation in the current MAXSC model. The threshold parameter, b, is assumed equal across all items, which could be too restrictive in many practical data analysis situations. Indeed, most of the IRT models for successive categories data (e.g., Muraki & Carlson, 1995; Samejima, 1969) that use a bilinear model (scalar products) in the response strength function as well as all factor analytic methods for discrete data (e.g., Christoffersson, 1975; Muthén, 1984) differentiate the threshold parameter for different items. Two exceptions are DeSarbo and Cho (1989) and Muraki (1990), although Muraki seems to have changed his mind (Muraki & Carlson, 1995). DeSarbo and his collaborators' models (DeSarbo & Cho, 1989; DeSarbo & Hoffman, 1986, 1987) as well as Takane's original proposal (1983) have the option of allowing b to vary over subjects, but this does not seem to be a wise choice, because it introduces another kind of incidental parameter. In any case, it is an interesting empirical question to investigate whether the threshold parameter really needs to be differentiated across items in particular situations. To do this, however, requires a computer program that can fit the model under both assumptions.

The current MAXSC is restricted to dichotomous data. It should not be too difficult to extend it to the general successive categories case, where the number as well as the nature of response categories may be different across different items. The possibility of differentiating the threshold parameter (or parameters) across different items will be more crucial in this case. The development of a computer program that can analyze such data is currently under way.

Takane (1984) discussed three kinds of categorical variables: unordered categorical variables; ordered categorical variables; and variables indicating subpopulations of subjects, such as gender, age group, and level of education. MAXMC and MAXSC, respectively, cover the first two kinds of categorical data. While the third kind of categorical variables can be analyzed as special cases of the first two, Takane also suggested an alternative approach. The data may be split into several sets (subpopulations) according to these variables, and each set may be modeled separately with the provision that some of the parameters in the separate models are constrained to be equal across different sets. For example, parameters in the choice model (3), may be assumed common, while the distribution of **y** may vary across different subpopulations. At the moment, no software exists that can accommodate all the three types of categorical data simultaneously. This will be an important undertaking in the near future.

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Appendix A. Statements in Andrich's (1988) data

1. Capital punishment is one of the most hideous practices of our time.
2. The state cannot teach the sacredness of human life by destroying it.
3. Capital punishment is not an effective deterrent to crime.
4. I don't believe in capital punishment but I am not sure it isn't necessary.
5. I think capital punishment is necessary but I wish it were not.
6. Until we find a more civilized way to prevent crime we must have capital punishment.
7. Capital punishment is justified because it does act as a deterrent to crime.
8. Capital punishment gives the criminal what he deserves.

Appendix B. Questions in Sugiyama's (1975) data

1. Do you make it a rule to practice religious conduct, such as attending religious services, religious worship and missionary works and do you occasionally offer prayers or chant sutras?
2. Do you visit a grave once or twice a year?
3. Do you occasionally read religious books, such as the Bible or the Buddhist Scriptures?
4. Do you visit shrines and temples to pray for business prosperity, success in an entrance examination and so forth?
5. Do you keep a talisman, such as an amulet, charm or mascot near you?
6. Do you draw a fortune, consult a diviner or have you had your fortune told within the last few years?