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LINEAR ALGEBRA AND ITS APPLICATIONS

On oblique projectors [1]

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Abstract

Oblique projectors are useful in many contexts, especially in the instrumental variable estimation in regression models in econometrics, where the disturbance term tends to be correlated with predictor variables. This paper addresses two important questions regarding oblique projectors: (1) products of two oblique projectors and (2) decompositions of oblique projectors. In (1), we investigate what determines the onto- and the along-spaces of an oblique projector defined as a product of two oblique projectors under a variety of conditions. In (2), we examine various decompositions of oblique projectors when both predictor variables and instrumental variables consist of two distinct sets of variables. These decompositions are analogous to those of orthogonal projectors.

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1. Introduction

Projectors play important roles in many statistical methods. Oblique projectors are particularly useful in instrumental variable estimation of regression models in econometrics, where the disturbance term tends to be correlated with predictor variables ([3], pp. 363–366). Let $\mathbf{P}_{V,W}$ represent the oblique projector onto V along W (i.e., $\text{Sp}(\mathbf{P}) = V$, where $\text{Sp}(\mathbf{P})$ denotes the range space of $\mathbf{P}$, and $\text{Ker}(\mathbf{P}) = W$, where $\text{Ker}(\mathbf{P})$ denotes the null space of $\mathbf{P}$. Let $\mathbf{Z}$ denote a matrix of predictor variables in regression, and $\mathbf{L}$ a matrix of instrumental variables, such that $\text{rank}(\mathbf{L}'\mathbf{Z}) = \text{rank}(\mathbf{Z}) = \text{rank}(\mathbf{L})$ (see Section 3 for the meaning of this rank requirement). Then, the instrumental variable estimation invokes a projection matrix onto $V = \text{Sp}(\mathbf{Z})$ along $W = \text{Ker}(\mathbf{L}')$.

This paper explains two important questions regarding oblique projectors. One is concerned with products of two oblique projectors, and the other with decompositions of oblique projectors. In the first part of this paper we investigate what determines the onto- and the along-spaces of a product of two oblique projectors under various conditions. In the second part we examine various decompositions of oblique projectors when both predictor variables and instrumental variables consist of two distinct sets of variables.

2. Products of two oblique projectors

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be two projection matrices of a same order. It is well known ([5], Theorem 5.1.4) that the commutativity of the two matrices provides a sufficient condition for the product of two projection matrices, $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$, to be also a projection matrix. However, as is well known, this is a sufficient but not a necessary condition. Brown and Page [1] provided a necessary and sufficient (ns) condition for $\mathbf{P}_1\mathbf{P}_2$ to be also a projection matrix. More recently, Groß and Trenkler [2] provided a number of interesting results on products of oblique projectors. In this paper we provide alternative (but often equivalent) characterizations of Groß and Trenkler's theorems. These characterizations shed further light on their theorems. In presenting our results we explicitly discuss their relations to Groß and Trenkler's results.

We first discuss equivalent conditions to Brown and Page's condition for $\mathbf{P}_1\mathbf{P}_2$ to be also a projection matrix, and give analogous conditions for each of $\mathbf{P}_1\mathbf{Q}_2$, $\mathbf{Q}_1\mathbf{P}_2$, and $\mathbf{Q}_1\mathbf{Q}_2$ to be also a projection matrix, where $\mathbf{Q}_i = \mathbf{I} - \mathbf{P}_i$, $i = 1, 2$. We then consider situations in which two or more of these conditions are simultaneously satisfied. It turns out that the commutative case arises when all the four conditions are simultaneously satisfied.

Let $\mathbf{P}_i$, $i = 1, 2$ be projectors onto space V_i along space W_i, where $V_i \oplus W_i = E^n$, $i = 1, 2$. We sometimes write this as $\mathbf{P}_i = \mathbf{P}_{V_i,W_i}$, $i = 1, 2$, since $\text{Sp}(\mathbf{P}_i) = V_i$ and $\text{Ker}(\mathbf{P}_i) = W_i$. Then, $\text{Sp}(\mathbf{Q}_i) = W_i$ and $\text{Ker}(\mathbf{Q}_i) = V_i$, $i = 1, 2$.

We define $\mathrm{Sp}(\mathbf{P}_1\mathbf{P}_2) = V_{12}$, $\mathrm{Ker}(\mathbf{P}_1\mathbf{P}_2) = W_{12}$, $\mathrm{Sp}(\mathbf{P}_2\mathbf{P}_1) = V_{21}$, and $\mathrm{Ker}(\mathbf{P}_2\mathbf{P}_1) = W_{21}$.

Lemma 1 (Condition (1)). *The following statements are equivalent*:
(i) $(\mathbf{P}_1\mathbf{P}_2)^2 = \mathbf{P}_1\mathbf{P}_2$.
(ii) $\mathbf{P}_1\mathbf{Q}_2\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$.
(iii) $\mathbf{P}_1\mathbf{Q}_2\mathbf{Q}_1\mathbf{P}_2 = \mathbf{0}$.
(iv) $\mathbf{P}_1\mathbf{P}_2\mathbf{Q}_1\mathbf{P}_2 = \mathbf{0}$.

Proofs of equivalences among these statements are rather trivial and will not be presented here. Note that the above condition implies $\mathrm{Ker}(\mathbf{P}_1\mathbf{P}_2) = \mathrm{Sp}(\mathbf{I} - \mathbf{P}_1\mathbf{P}_2)$, so that $V_{12} \oplus W_{12} = E^n$.

Note 1. Equivalence between (1-i) and $V_{12} \subset V_2 \oplus (W_1 \cap W_2)$ has been pointed out by Brown and Page ([1], p. 339) without a proof. This has recently been proved by Groß and Trenkler [2], who used equivalences between (1-i) and (ii), and the fact that (1-ii) implied $\mathrm{Sp}(\mathbf{Q}_2\mathbf{P}_1\mathbf{P}_2) \subset W_1 \cap W_2$ as part of their proof. Groß and Trenkler also state that under Condition (1), $\mathrm{Sp}(\mathbf{P}_1\mathbf{P}_2) = V_1 \cap (V_2 \oplus (W_1 \cap W_2))$ and $\mathrm{Sp}(\mathbf{I} - \mathbf{P}_1\mathbf{P}_2) = \mathrm{Ker}(\mathbf{P}_1\mathbf{P}_2) = W_2 \oplus (W_1 \cap V_2)$. (Although $\mathrm{Ker}(\mathbf{P}_1\mathbf{P}_2) = W_2 \oplus (W_1 \cap V_2)$ holds in general, $\mathrm{Sp}(\mathbf{I} - \mathbf{P}_1\mathbf{P}_2) = \mathrm{Ker}(\mathbf{P}_1\mathbf{P}_2)$ holds if and only if Condition (1) holds.)

Note 2. Werner ([8], Lemma 2.2) has shown that an ns condition for (1-i) is $V_2 \subset V_1 \oplus (W_1 \cap V_2) \oplus (W_1 \cap W_2)$. A proof somewhat simpler than Werner's is given below, which uses some of the relations stated Lemma 1.

Assume (1-i) and let $\mathbf{x} \in V_2$. Then, $\mathbf{P}_2\mathbf{x} = \mathbf{x}$, and $\mathbf{x} = \mathbf{P}_1\mathbf{x} + \mathbf{P}_2\mathbf{Q}_1\mathbf{x} + \mathbf{Q}_2\mathbf{Q}_1\mathbf{x} = \mathbf{P}_1\mathbf{x} + \mathbf{P}_2\mathbf{Q}_1\mathbf{P}_2\mathbf{x} + \mathbf{Q}_2\mathbf{Q}_1\mathbf{P}_2\mathbf{x}$. However, $\mathbf{P}_1\mathbf{x} \in V_1$, $\mathbf{P}_2\mathbf{Q}_1\mathbf{P}_2\mathbf{x} \in W_1 \cap V_2$ because of (1-iv), and $\mathbf{Q}_2\mathbf{Q}_1\mathbf{P}_2\mathbf{x} \in W_1 \cap W_2$ because of (1-iii), implying $V_2 \subset V_1 \oplus (W_1 \cap V_2) \oplus (W_1 \cap W_2)$.

Conversely, let $\mathbf{x} \in E^n$. Then, $\mathbf{P}_2\mathbf{x} \in V_2$. Let $\mathbf{P}_2\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3$, where $\mathbf{x}_1 \in V_1$, $\mathbf{x}_2 \in W_1 \cap V_2$, and $\mathbf{x}_3 \in W_1 \cap W_2$. We have $(\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1)\mathbf{P}_2\mathbf{x} = \mathbf{P}_1\mathbf{P}_2\mathbf{x}_1 = \mathbf{P}_1\mathbf{P}_2(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) = \mathbf{P}_1\mathbf{P}_2\mathbf{x}$, implying (1-i).

Note 3. In Lemma 1, $\mathbf{P}_1\mathbf{Q}_2$ and $\mathbf{Q}_1\mathbf{P}_2$ are not necessarily projection matrices.

Note 4. We can derive conditions similar to Condition (1) for $\mathbf{P}_1\mathbf{Q}_2, \mathbf{Q}_1\mathbf{P}_2$ and $\mathbf{Q}_1\mathbf{Q}_2$ to be also projection matrices, which we state as Corollaries 1–3:

Corollary 1 (Condition (2)). *The following statements are equivalent*:
(i) $(\mathbf{P}_1\mathbf{Q}_2)^2 = \mathbf{P}_1\mathbf{Q}_2$.
(ii) $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1\mathbf{Q}_2 = \mathbf{0}$.
(iii) $\mathbf{P}_1\mathbf{P}_2\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{0}$.
(iv) $\mathbf{P}_1\mathbf{Q}_2\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{0}$.

Corollary 2 (Condition (3)). *The following statements are equivalent*:
(i) $(\mathbf{Q}_1\mathbf{P}_2)^2 = \mathbf{Q}_1\mathbf{P}_2$.
(ii) $\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_1\mathbf{P}_2 = \mathbf{0}$.
(iii) $\mathbf{Q}_1\mathbf{Q}_2\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$.
(iv) $\mathbf{Q}_1\mathbf{P}_2\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$.

Corollary 3 (Condition (4)). *The following statements are equivalent*:
(i) $(\mathbf{Q}_1\mathbf{Q}_2)^2 = \mathbf{Q}_1\mathbf{Q}_2$.
(ii) $\mathbf{Q}_1\mathbf{P}_2\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{0}$.
(iii) $\mathbf{Q}_1\mathbf{P}_2\mathbf{P}_1\mathbf{Q}_2 = \mathbf{0}$.
(iv) $\mathbf{Q}_1\mathbf{Q}_2\mathbf{P}_1\mathbf{Q}_2 = \mathbf{0}$.

Note 5. According to Groß and Trenkler's result mentioned in Note 1, $\mathrm{Sp}(\mathbf{P}_1\mathbf{Q}_2) = V_1 \cap (W_2 \oplus (W_1 \cap V_2))$, and $\mathrm{Ker}(\mathbf{P}_1\mathbf{Q}_2) = V_2 \oplus (W_1 \cap W_2)$ under Condition (2), $\mathrm{Sp}(\mathbf{Q}_1\mathbf{P}_2) = W_1 \cap (V_2 \oplus (V_1 \cap W_2))$, and $\mathrm{Ker}(\mathbf{Q}_1\mathbf{P}_2) = W_2 \oplus (V_1 \cap V_2)$ under Condition (3), and $\mathrm{Sp}(\mathbf{Q}_1\mathbf{Q}_2) = W_1 \cap (W_2 \oplus (V_1 \cap V_2))$, and $\mathrm{Ker}(\mathbf{Q}_1\mathbf{Q}_2) = V_2 \oplus (V_1 \cap W_2)$ under Condition (4).

Note 6. Conditions analogous to Conditions (1)–(4) can also be given for $\mathbf{P}_2\mathbf{P}_1$, $\mathbf{Q}_2\mathbf{P}_1$, $\mathbf{P}_2\mathbf{Q}_1$, and $\mathbf{Q}_2\mathbf{Q}_1$, which we call Conditions (1′)–(4′), respectively. Note that $V_{21} = V_2 \cap (V_1 \oplus (W_1 \cap W_2))$, and $W_{21} = W_1 \oplus (V_1 \cap W_2)$ under Condition (1′), $\mathrm{Sp}(\mathbf{Q}_2\mathbf{P}_1) = W_2 \cap (V_1 \oplus (W_1 \cap V_2))$, and $\mathrm{Ker}(\mathbf{Q}_2\mathbf{P}_1) = W_1 \oplus (V_1 \cap V_2)$ Funder Condition (2′), $\mathrm{Sp}(\mathbf{P}_2\mathbf{Q}_1) = V_2 \cap (W_1 \oplus (V_1 \cap W_2))$, and $\mathrm{Ker}(\mathbf{P}_2\mathbf{Q}_1) = V_1 \oplus (W_1 \cap W_2)$ under Condition (3′), and $\mathrm{Sp}(\mathbf{Q}_2\mathbf{Q}_1) = W_2 \cap (W_1 \oplus (V_1 \cap V_2))$, and $\mathrm{Ker}(\mathbf{Q}_2\mathbf{Q}_1) = V_1 \oplus (W_1 \cap V_2)$ under Condition (4′).

The following lemma due to Groß and Trenkler ([2] Lemma 2) is useful in the sequel.

Lemma 2.
(a) $V_1 \cap V_2 \subset V_{12}$ *and* $V_1 \cap V_2 \subset V_{21}$.
(b) $W_{12} \subset W_1 + W_2$ *and* $W_{21} \subset W_1 + W_2$.

Lemma 2 concerns only those relations pertaining to V_{12}, W_{12}, V_{21} and W_{21}. Similar relations can be stated for the onto- and the along-spaces of $\mathbf{P}_1\mathbf{Q}_2$, $\mathbf{Q}_2\mathbf{P}_1$, $\mathbf{Q}_1\mathbf{P}_2$, $\mathbf{P}_2\mathbf{Q}_1$, $\mathbf{Q}_1\mathbf{Q}_2$, and $\mathbf{Q}_2\mathbf{Q}_1$ as well.

Interesting special cases follow from concatenating two conditions at a time among the four conditions (Conditions (1)–(4)) discussed above. We discuss combinations of Conditions (1) and (2), Conditions (1) and (3), Conditions (2) and (4), and Conditions (3) and (4). The other two possible combinations, Conditions (1) and (4) and Conditions (2) and (3), do not seem to lead to any interesting cases.

Theorem 1 (Condition (5)). *The following statements are equivalent*:

(i) *Conditions* (1) *and* (2).

(ii) $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_2$.

(iii) $\mathbf{P}_1\mathbf{P}_2\mathbf{Q}_1 = \mathbf{0}$ $(\mathrm{Sp}(\mathbf{P}_2\mathbf{Q}_1) \subset W_1 \subset W_{12})$.

(iv) $W_{12} = W_1 + W_2$.

(v) $\mathrm{Sp}(\mathbf{P}_2\mathbf{Q}_1) = W_1 \cap V_2$.

(vi) *Conditions* (3′) *and* (4′).

(vii) $\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_1 = \mathbf{Q}_2\mathbf{Q}_1$.

(viii) $\mathbf{P}_1\mathbf{Q}_2\mathbf{Q}_1 = \mathbf{0}$ $(\mathrm{Sp}(\mathbf{Q}_2\mathbf{Q}_1) \subset W_1 \subset \mathrm{Ker}(\mathbf{P}_1\mathbf{Q}_2))$.

(ix) $\mathrm{Ker}(\mathbf{P}_1\mathbf{Q}_2) = W_1 + V_2$.

(x) $\mathrm{Sp}(\mathbf{Q}_2\mathbf{Q}_1) = W_1 \cap W_2$.

Proof. The 10 statements in Theorem 1 can be grouped into two, (i)–(v) in one group and (vi)–(x) in the other. We first show equivalences among statements in each group, and then show equivalence between the two groups by showing one of the statements in the first group (iii) is equivalent to one of the statements in the second group (viii).

To show equivalence between (i) and (iii), we add both sides of (1-iv) and (2-iii) and obtain (iii). Conversely, if (iii) holds, both (1-iv) and (2-iii) are trivially true. Statement (ii) is just a restatement of (iii). To show equivalence between (iii) and (iv), we first note $W_2 \subset W_{12}$ which holds in general, but since $W_1 \subset W_{12}$ from (5-iii), we obtain $W_1 + W_2 \subset W_{12}$. We also have $W_{12} \subset W_1 + W_2$ from Lemma (2-b). Together these imply $W_{12} = W_1 + W_2$. Equivalence between (iii) and (v) can be shown in a similar manner.

It is obvious that (i) and (vi) are parallel statements, so are (ii) and (vii), (iii) and (viii), and so on, so that equivalences among (vi)–(x) can be proved in a manner analogous to the above. Finally, we have (5-iii) if and only if (5-viii), since $\mathbf{P}_1\mathbf{P}_2\mathbf{Q}_1 = \mathbf{0}$ holds if and only if $\mathbf{P}_1(\mathbf{I} - \mathbf{Q}_2)\mathbf{Q}_1 = \mathbf{0}$, which in turn holds if and only if $\mathbf{P}_1\mathbf{Q}_2\mathbf{Q}_1 = \mathbf{0}$, concluding the proof. □

Note 7. Statements (5-iv), (5-v), (5-ix) and (5-x) imply $W_2 \oplus (W_1 \cap V_2) = W_1 + W_2$, $V_2 \cap (W_1 \oplus (V_1 \cap W_2)) = W_1 \cap V_2$, $V_2 \oplus (W_1 \cap W_2) = W_1 + V_2$, and $W_2 \cap (W_1 \oplus (V_1 \cap V_2)) = W_1 \cap W_2$, respectively.

Similarly we have, the following corollary.

Corollary 4 (Condition (6)). *The following statements are equivalent*:

(i) *Conditions* (1) *and* (3).

(ii) $\mathbf{P}_2\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2$.

(iii) $\mathbf{Q}_2\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$ $(V_{12} \subset V_2 \subset \mathrm{Ker}(\mathbf{Q}_2\mathbf{P}_1))$.

(iv) $V_{12} = V_1 \cap V_2$.

(v) $\mathrm{Ker}(\mathbf{Q}_2\mathbf{P}_1) = W_1 + V_2$.

(vi) *Conditions* (2′) *and* (4′).

(vii) $\mathbf{Q}_2\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{Q}_2\mathbf{Q}_1$.
(viii) $\mathbf{Q}_2\mathbf{Q}_1\mathbf{P}_2 = \mathbf{0}$ $(\mathrm{Sp}(\mathbf{Q}_1\mathbf{P}_2) \subset V_2 \subset \mathrm{Ker}(\mathbf{Q}_2\mathbf{Q}_1))$.
(ix) $\mathrm{Sp}(\mathbf{Q}_1\mathbf{P}_2) = W_1 \cap V_2$.
(x) $\mathrm{Ker}(\mathbf{Q}_2\mathbf{Q}_1) = V_1 + V_2$.

Note 8. Statements (6-iv), (6-v), (6-ix), and (6-x) imply $V_1 \cap (V_2 \oplus (W_1 \cap W_2)) = V_1 \cap V_2$, $W_1 \oplus (V_1 \cap V_2) = W_1 + V_2$, $W_1 \cap (V_2 \oplus (V_1 \cap W_2)) = W_1 \cap V_2$, and $V_1 \oplus (W_1 \cap V_2) = V_1 + V_2$, respectively.

Corollary 5 (Condition (7)). *The following statements are equivalent*:
(i) *Conditions* (2) *and* (4).
(ii) $\mathbf{P}_2\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$.
(iii) $\mathbf{P}_2\mathbf{P}_1\mathbf{Q}_2 = \mathbf{0}$ $(\mathrm{Sp}(\mathbf{P}_1\mathbf{Q}_2) \subset W_2 \subset W_{21})$.
(iv) $W_{21} = W_1 + W_2$.
(v) $\mathrm{Sp}(\mathbf{P}_1\mathbf{Q}_2) = V_1 \cap W_2$.
(vi) *Conditions* (1′) *and* (3′).
(vii) $\mathbf{Q}_2\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{Q}_1\mathbf{Q}_2$.
(viii) $\mathbf{P}_2\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{0}$ $(\mathrm{Sp}(\mathbf{Q}_1\mathbf{Q}_2) \subset W_2 \subset \mathrm{Ker}(\mathbf{P}_2\mathbf{Q}_1))$.
(ix) $\mathrm{Sp}(\mathbf{Q}_1\mathbf{Q}_2) = W_1 \cap W_2$.
(x) $\mathrm{Ker}(\mathbf{P}_2\mathbf{Q}_1) = V_1 + W_2$.

Note 9. Statements (7-iv), (7-v), (7-ix), and (7-x) imply that $W_1 \oplus (V_1 \cap W_2) = W_1 + W_2$, $V_1 \cap (W_2 \oplus (W_1 \cap V_2)) = V_1 \cap W_2$, $W_1 \cap (W_2 \oplus (V_1 \cap V_2)) = W_1 \cap W_2$, and $V_1 \oplus (W_1 \cap W_2) = V_1 + W_2$, respectively.

Corollary 6 (Condition (8)). *The following statements are equivalent*:
(i) *Conditions* (3) *and* (4).
(ii) $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_2\mathbf{P}_1$.
(iii) $\mathbf{Q}_1\mathbf{P}_2\mathbf{P}_1 = \mathbf{0}$ $(V_{21} \subset V_1 \subset \mathrm{Ker}(\mathbf{Q}_1\mathbf{P}_2))$.
(iv) $V_{21} = V_1 \cap V_2$.
(v) $\mathrm{Ker}(\mathbf{Q}_1\mathbf{P}_2) = V_1 + W_2$.
(vi) *Conditions* (1′) *and* (2′).
(vii) $\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_1 = \mathbf{Q}_1\mathbf{Q}_2$.
(viii) $\mathbf{Q}_1\mathbf{Q}_2\mathbf{P}_1 = \mathbf{0}$ $(\mathrm{Sp}(\mathbf{Q}_2\mathbf{P}_1) \subset V_1 \subset \mathrm{Ker}(\mathbf{Q}_1\mathbf{Q}_2))$.
(ix) $\mathrm{Ker}(\mathbf{Q}_1\mathbf{Q}_2) = V_1 + V_2$.
(x) $\mathrm{Sp}(\mathbf{Q}_2\mathbf{P}_1) = V_1 \cap W_2$.

Note 10. Statements (8-iv), (8-v), (8-ix), and (8-x) imply that $V_2 \cap (V_1 \oplus (W_1 \cap W_2)) = V_1 \cap V_2$, $W_2 \oplus (V_1 \cap V_2) = V_1 + W_2$, $V_2 \oplus (V_1 \cap W_2) = V_1 + V_2$, and $W_2 \cap (V_1 \oplus (W_1 \cap V_2)) = V_1 \cap W_2$, respectively.

Proofs of equivalences among statements within Corollaries 4–6 are similar to those for Theorem 1 and will not be given here.

We now take three conditions at a time from Conditions (1)–(4). We consider all possible combinations of three conditions at a time, namely

Conditions (1)–(3), Conditions (1), (2), and (4), Conditions (1), (3), and (4), and Conditions (2)–(4).

Theorem 2 (Condition (9)). *The following statements are equivalent*:

(i) *Conditions* (1)–(3).

(ii) $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_2\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2$.

(iii) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_{V_1 \cap V_2, W_1+W_2}$, *where the latter is the projection matrix onto* $V_1 \cap V_2$ *along* $W_1 + W2$.

(iv) *Conditions* (2′)–(4′).

(v) $\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_1 = \mathbf{Q}_2\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{Q}_2\mathbf{Q}_1$.

(vi) $\mathbf{Q}_2\mathbf{Q}_1 = \mathbf{P}_{W_1 \cap W_2, V_1+V_2}$.

Proof. As in the case of the statements in Theorem 1, the six statements in Theorem 2 can be grouped into two distinct groups, one consisting of (i)–(iii), and the other (iv)–(vi). We show equivalences among statements within groups, and then between groups.

We obtain (i) by the combination of Conditions (5) and (6). We also obtain (ii) from (5-ii) and (6-ii), establishing equivalence between (i) and (ii). Equivalence between (i) and (iii) can be seen by noting (5-iv) and (6-iv). (Although the latter was not shown explicitly in Corollary 4, it can easily be shown in a manner similar to that for (5-iv) in Theorem 1.)

Equivalences among (iv)–(vi) can be shown in a manner parallel to the above. Finally, (9-i) and (9-iv) are equivalent, because Conditions (1) and (2) are equivalent to Conditions (3′) and (4′) according to Theorem 1, and Conditions (1) and (3) and Conditions (2′) and (4′) are equivalent according to Corollary 4, establishing equivalences among all the statements in Theorem 2.

Note that under Condition (9), $(V_1 \cap V_2) \oplus W_1 + W_2 = E^n$ and $(W_1 \cap W_2) \oplus V_1 + V_2 = E^n$. □

Note 11. Theorem 3 of Groß and Trenkler [2] states that ns conditions for (9-iii) are (1) $V_1 + V_2 = V_1 \oplus (W_1 \cap V_2)$, and (2) $V_1 + V_2 \oplus (W_1 \cap W_2) = E^n$. To understand the relation between Theorem 3 of Groß and Trenkler and our Theorem 2, note (9-vi) above, which holds if and only if (9-iv) holds. Under Condition (4′), $\mathbf{Q}_2\mathbf{Q}_1$ is a projector along $V_1 \oplus (W_1 \cap V_2)$ (see Note 6), which reduces to $V_1 + V_2$ under Conditions (2′) and (4′), implying (1). Also, under Conditions (3′) and (4′) $\mathrm{Sp}(\mathbf{Q}_2\mathbf{Q}_1) = W_1 \cap W_2$. Thus, under Conditions (2′)–(4′), (2) must hold. Conversely, if (1) and (2) hold, (9-iii) must hold (according to Theorem 3 of Groß and Trenkler), which holds if and only if Conditions (2′), (3′) and (4′) hold (according to our Theorem 2). Thus, the two theorems are equivalent. It is interesting to note that in Groß and Trenkler's theorem, the condition for $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_{V_1 \cap V_2, W_1+W_2}$ is stated in terms of conditions on the onto- and the along-spaces of $\mathbf{Q}_2\mathbf{Q}_1$. The condition equivalent to (1) and (2) can be stated in terms of conditions on the onto- and the along-space of $\mathbf{P}_1\mathbf{P}_2$, which are (1′) $W_1 + W_2 = W_2 \oplus (W_1 \cap V_2)$, and (2′) $W_1 + W_2 \oplus (V_1 \cap V_2) = E^n$.

Note 12. Under Condition (9), $\mathrm{Sp}(\mathbf{P}_2\mathbf{Q}_1) = \mathrm{Sp}(\mathbf{Q}_1\mathbf{P}_2) = W_1 \cap V_2$, and $\mathrm{Ker}(\mathbf{P}_1\mathbf{Q}_2) = \mathrm{Ker}(\mathbf{Q}_2\mathbf{P}_1) = W_1 + V_2$.

Note that Condition (9) only states a set of equivalent conditions under which $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_{V_1\cap V_2, W_1+W_2}$. Nothing is said about $\mathbf{P}_2\mathbf{P}_1$, which may or may not even be a projector. Indeed, as will be shown later (Theorem 3), we need all four conditions (Conditions (1)–(4)) to establish $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_{V_1\cap V_2, W_1+W_2}$.

Similarly for Theorem 2, we have

Corollary 7 (Condition (10)). *The following statements are equivalent*:
(i) *Conditions* (1), (2) *and* (4).
(ii) $\mathbf{P}_1\mathbf{Q}_2\mathbf{P}_1 = \mathbf{Q}_2\mathbf{P}_1\mathbf{Q}_2 = \mathbf{P}_1\mathbf{Q}_2 = \mathbf{P}_{V_1\cap W_2, W_1+V_2}$.
(iii) *Conditions* (1′), (3′) *and* (4′).
(iv) $\mathbf{P}_2\mathbf{Q}_1\mathbf{P}_2 = \mathbf{Q}_1\mathbf{P}_2\mathbf{Q}_1 = \mathbf{P}_2\mathbf{Q}_1 = \mathbf{P}_{W_1\cap V_2, V_1+W_2}$.

Note 13. Under Condition (10), $\mathrm{Sp}(\mathbf{Q}_2\mathbf{Q}_1) = \mathrm{Sp}(\mathbf{Q}_1\mathbf{Q}_2) = W_1 \cap W_2$, and $\mathrm{Ker}(\mathbf{P}_1\mathbf{P}_2) = \mathrm{Ker}(\mathbf{P}_2\mathbf{P}_1) = W_1 + W_2$.

Corollary 8 (Condition (11)). *The following statements are equivalent*:
(i) *Conditions* (1), (3) *and* (4).
(ii) $\mathbf{Q}_1\mathbf{P}_2\mathbf{Q}_1 = \mathbf{P}_1\mathbf{Q}_2\mathbf{P}_1 = \mathbf{Q}_1\mathbf{P}_2 = \mathbf{P}_{W_1\cap V_2, V_1+W_2}$.
(iii) *Conditions* (1′), (2′) *and* (4′).
(iv) $\mathbf{Q}_2\mathbf{P}_1\mathbf{Q}_2 = \mathbf{P}_1\mathbf{Q}_2\mathbf{P}_1 = \mathbf{Q}_2\mathbf{P}_1 = \mathbf{P}_{V_1\cap W_2, W_1+V_2}$.

Note 14. Under Condition (11), $\mathrm{Sp}(\mathbf{P}_1\mathbf{P}_2) = \mathrm{Sp}(\mathbf{P}_2\mathbf{P}_1) = V_1 \cap V_2$, and $\mathrm{Ker}(\mathbf{Q}_2\mathbf{Q}_1) = \mathrm{Ker}(\mathbf{Q}_1\mathbf{Q}_2) = V_1 + V_2$.

Corollary 9 (Condition (12)). *The following statements are equivalent*:
(i) *Conditions* (2)–(4).
(ii) $\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_1 = \mathbf{Q}_2\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{Q}_1\mathbf{Q}_2 = \mathbf{P}_{W_1\cap W_2, V_1+V_2}$.
(iii) *Conditions* (1′)–(3′).
(iv) $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_2\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_{V_1\cap V_2, W_1+W_2}$.

Note 15. Under Condition (12), $\mathrm{Sp}(\mathbf{P}_1\mathbf{Q}_2) = \mathrm{Sp}(\mathbf{Q}_2\mathbf{P}_1) = V_1 \cap W_2$, and $\mathrm{Ker}(\mathbf{P}_2\mathbf{Q}_1) = \mathrm{Ker}(\mathbf{Q}_1\mathbf{P}_2) = V_1 + W_2$.

Proofs of equivalences among statements within Corollaries 7–9 are all similar to that for Theorem 2, and will not be given here.

We now assume that all of Conditions (1)–(4) hold. This leads to the following characterization of the commutativity of $\mathbf{P}_1$ and $\mathbf{P}_2$, different from that of Rao and Mitra ([5], ch. 5).

Theorem 3 (Condition (13)). *The following statements are equivalent*:
(i) *Conditions* (1)–(4).
(ii) *Conditions* (1′)–(4′).
(iii) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_2\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_{V_1\cap V_2, W_1+W_2}$.
(iv) $\mathbf{P}_1\mathbf{Q}_2 = \mathbf{P}_1\mathbf{Q}_2\mathbf{P}_1 = \mathbf{Q}_2\mathbf{P}_1\mathbf{Q}_2 = \mathbf{Q}_2\mathbf{P}_1 = \mathbf{P}_{V_1\cap W_2, W_1+V_2}$.
(v) $\mathbf{Q}_1\mathbf{P}_2 = \mathbf{Q}_1\mathbf{P}_2\mathbf{Q}_1 = \mathbf{P}_2\mathbf{Q}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{Q}_1 = \mathbf{P}_{W_1\cap V_2, V_1+W_2}$.
(vi) $\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_1 = \mathbf{Q}_2\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{Q}_2\mathbf{Q}_1 = \mathbf{P}_{W_1\cap W_2, V_1+V_2}$.

Proof. Equivalences among (iii)–(vi) are well known. Equivalences among (i), (ii) and (iii) are trivial, since Conditions (1) and (2) are equivalent to (3′) and (4′) and Conditions (3) and (4) are equivalent to (1′) and (2′) (see Conditions (5) and (8)). □

Note 16. Theorem 4 of Groß and Trenkler [2] states that ns conditions for (13-iii) are (1) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_{V_1\cap V_2, W_1+W_2}$, and (2) $\mathrm{rank}(\mathbf{P}_1\mathbf{P}_2) = \mathrm{rank}(\mathbf{P}_2\mathbf{P}_1)$, and their Remark 4 states that the latter condition can be replaced by (2′) $V_{21} \subset V_1$. Note that (1) is equivalent to Conditions (1)–(3) (our Theorem 2), and (2′) is equivalent to Conditions (3) and (4) (see (8-iii)). Jointly they are equivalent to Condition (13). Note also that Conditions (9) and Conditions (2) and (4) are also jointly equivalent to Condition (13). This implies that (2′) can also be replaced by $W_2 \subset W_{21}$ (see (7-iii)).

Note 17. Under Condition (13), we have (1) $\mathrm{rank}(\mathbf{P}_1\mathbf{P}_2) + \mathrm{rank}(\mathbf{P}_1\mathbf{Q}_2) + \mathrm{rank}(\mathbf{Q}_1\mathbf{P}_2) + \mathrm{rank}(\mathbf{Q}_1\mathbf{Q}_2) = n$, and (2) $\mathrm{rank}(\mathbf{P}_2\mathbf{P}_1) + \mathrm{rank}(\mathbf{Q}_2\mathbf{P}_1) + \mathrm{rank}(\mathbf{P}_2\mathbf{Q}_1) + \mathrm{rank}(\mathbf{Q}_2\mathbf{Q}_1) = n$ [4]. This means that E^n is decomposed into $E^n = (V_1 \cap V_2) \oplus (V_1 \cap W_2) \oplus (W_1 \cap V_2) \oplus (W_1 \cap W_2)$, where $V_1 \cap V_2 = \mathrm{Sp}(\mathbf{P}_1\mathbf{P}_2) = \mathrm{Sp}(\mathbf{P}_2\mathbf{P}_1)$, $V_1 \cap W_2 = \mathrm{Sp}(\mathbf{P}_1\mathbf{Q}_2) = \mathrm{Sp}(\mathbf{Q}_2\mathbf{P}_1)$, $W_1 \cap V_2 = \mathrm{Sp}(\mathbf{Q}_1\mathbf{P}_2) = \mathrm{Sp}(\mathbf{P}_2\mathbf{Q}_1)$, and $W_1 \cap W_2 = \mathrm{Sp}(\mathbf{Q}_1\mathbf{Q}_2) = \mathrm{Sp}(\mathbf{Q}_2\mathbf{Q}_1)$, which imply $\mathrm{rank}(\mathbf{P}_1\mathbf{P}_2) = \mathrm{rank}(\mathbf{P}_2\mathbf{P}_1)$, $\mathrm{rank}(\mathbf{P}_1\mathbf{Q}_2) = \mathrm{rank}(\mathbf{Q}_2\mathbf{P}_1)$, $\mathrm{rank}(\mathbf{Q}_1\mathbf{P}_2) = \mathrm{rank}(\mathbf{P}_2\mathbf{Q}_1)$, and $\mathrm{rank}(\mathbf{Q}_1\mathbf{Q}_2) = \mathrm{rank}(\mathbf{Q}_2\mathbf{Q}_1)$.

3. Decompositions of oblique projectors

The second problem we deal with relates to decompositions of oblique projectors. In this section we examine various decompositions of oblique projectors when both predictor variables and instrumental variables consist of two distinct sets of variables. Let $\mathbf{Z}$ be a matrix of predictor variables in regression, and let $\mathbf{L}$ be a matrix of instrumental variables. Depending on the relationships between the two matrices, various decompositions of $\mathbf{P}_{\mathrm{Sp}(Z),\mathrm{Ker}(L')}$ are possible. For notational convenience, we denote $\mathbf{P}_{\mathrm{Sp}(Z),\mathrm{Ker}(L')}$ by $\mathbf{P}_{Z:L}$. We assume throughout this section that

Condition (14). $\text{rank}(\mathbf{L}'\mathbf{Z}) = \text{rank}(\mathbf{Z}) = \text{rank}(\mathbf{L})$.

Note that this condition is equivalent to $\text{Sp}(\mathbf{Z}) \oplus \text{Ker}(\mathbf{L}') = E^n$. Note also that under $\text{rank}(\mathbf{L}'\mathbf{Z}) = \text{rank}(\mathbf{Z})$, $(\mathbf{L}'\mathbf{Z})^-\mathbf{L}' \in \{\mathbf{Z}^-\}$, and under $\text{rank}(\mathbf{L}'\mathbf{Z}) = \text{rank}(\mathbf{L})$, $\mathbf{Z}(\mathbf{L}'\mathbf{Z})^- \in \{(\mathbf{L}')^-\}$ ([5], Section 4.11). Under (14), $\mathbf{P}_{Z:L} = \mathbf{Z}(\mathbf{L}'\mathbf{Z})^-\mathbf{L}'$ is the projector onto $\text{Sp}(\mathbf{Z})$ along $\text{Ker}(\mathbf{L}')$ [9]. Let $\mathbf{Z}$ be partitioned into $[\mathbf{X}|\mathbf{Y}]$, and let $\mathbf{L}$ be analogously partitioned into $[\mathbf{M}|\mathbf{N}]$. We assume, analogously to Condition (14), that

Condition (15). $\text{rank}(\mathbf{M}'\mathbf{X}) = \text{rank}(\mathbf{X}) = \text{rank}(\mathbf{M})$,

Condition (16). $\text{rank}(\mathbf{N}'\mathbf{Y}) = \text{rank}(\mathbf{Y}) = \text{rank}(\mathbf{N})$.

Decompositions of $\mathbf{P}_{Z:L}$ we will discuss are mostly analogous to those of orthogonal projectors. To motivate these decompositions we briefly review representative decompositions of orthogonal projectors. Let $\mathbf{Z} = \mathbf{L}$. We then have $\mathbf{P}_Z = \mathbf{P}_{Z:Z} = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^-\mathbf{Z}'$. Define $\mathbf{Q}_Z = \mathbf{I} - \mathbf{P}_Z$. Then, $\mathbf{P}_Z^2 = \mathbf{P}_Z$, $\mathbf{Q}_Z^2 = \mathbf{Q}_Z$, $\mathbf{P}'_Z = \mathbf{P}_Z$, $\mathbf{Q}'_Z = \mathbf{Q}_Z$, and $\mathbf{P}_Z\mathbf{Q}_Z = \mathbf{Q}_Z\mathbf{P}_Z = \mathbf{0}$. When $\mathbf{Z}$ is partitioned, various decompositions of $\mathbf{P}_Z$ are possible, as given below. The first four of them were mentioned in Ref. [6], and the last one was first noted by Takane et al. [7] and was presented, in its general form, in Ref. [10].

Lemma 3.
(i) $\mathbf{P}_Z = \mathbf{P}_X + \mathbf{P}_Y$ *if and only if* $\mathbf{X}'\mathbf{Y} = \mathbf{0}$.
(ii) $\mathbf{P}_Z = \mathbf{P}_X + \mathbf{P}_Y - \mathbf{P}_X\mathbf{P}_Y$ *if and only if* $\mathbf{P}_X\mathbf{P}_Y = \mathbf{P}_Y\mathbf{P}_X$.
(iii) $\mathbf{P}_Z = \mathbf{P}_X + \mathbf{P}_{Q_XY} = \mathbf{P}_Y + \mathbf{P}_{Q_YX}$.
(iv) $\mathbf{P}_Z = \mathbf{P}_{X/Q_Y} + \mathbf{P}_{Y/Q_X}$ *if and only if* $\text{Sp}(\mathbf{X}) \cap \text{Sp}(\mathbf{Y}) = \{\mathbf{0}\}$, *where* $\mathbf{P}_{X/Q_Y} = \mathbf{X}(\mathbf{X}'\mathbf{Q}_Y\mathbf{X})^-\mathbf{X}'\mathbf{Q}_Y$ *and* $\mathbf{P}_{Y/Q_X} = \mathbf{Y}(\mathbf{Y}'\mathbf{Q}_X\mathbf{Y})^-\mathbf{Y}'\mathbf{Q}_X$.
(v) $\mathbf{P}_Z = \mathbf{P}_{ZA} + \mathbf{P}_{Z(Z'Z)^-B}$, *where* $\text{Sp}(\mathbf{A}) = \text{Ker}(\mathbf{B}')$ *and* $\mathbf{Z}'\mathbf{W} = \mathbf{B}$ *for some* $\mathbf{W}$.

Note 18. (1) Holds if and only if $\mathbf{X}$ and $\mathbf{Y}$ are mutually orthogonal. Note that this condition is also equivalent to $\mathbf{P}_X\mathbf{P}_Y = \mathbf{P}_Y\mathbf{P}_X = \mathbf{0}$.

(2) Holds when $\mathbf{P}_X$ and $\mathbf{P}_Y$ may not be orthogonal, but commutative. This implies that the part of $\text{Sp}(\mathbf{X})$ and the part of $\text{Sp}(\mathbf{Y})$ excluding their common space, $\text{Sp}(\mathbf{X}) \cap \text{Sp}(\mathbf{Y})$, are mutually orthogonal. That is, $(\mathbf{P}_X - \mathbf{P}_X\mathbf{P}_Y)(\mathbf{P}_Y - \mathbf{P}_X\mathbf{P}_Y) = (\mathbf{P}_Y - \mathbf{P}_X\mathbf{P}_Y)(\mathbf{P}_X - \mathbf{P}_X\mathbf{P}_Y) = \mathbf{0}$. This decomposition plays an important role in two-way ANOVA with no interaction. Let $\mathbf{X}$ and $\mathbf{Y}$ denote matrices of dummy variables for the two factors. Then, $\mathbf{P}_X\mathbf{P}_Y = \mathbf{P}_Y\mathbf{P}_X = \mathbf{P}_{1_n}$, where $\mathbf{1}_n$ is the n-component vector of ones. We have $\mathbf{P}_Z - \mathbf{P}_{1_n} = (\mathbf{P}_X - \mathbf{P}_{1_n}) + (\mathbf{P}_Y - \mathbf{P}_{1_n})$, where the first term on the right hand side of this equation pertains to the main effect of $\mathbf{X}$, and the second term to the main effect of $\mathbf{Y}$.

(3) Holds in general. This decomposition is useful when we fit one of $\mathbf{X}$ and $\mathbf{Y}$ first, and then fit the other to the residual in multiple regression analysis. We have $\mathbf{P}_X\mathbf{P}_{Q_XY} = \mathbf{P}_{Q_XY}\mathbf{P}_X = \mathbf{0}$ and $\mathbf{P}_Y\mathbf{P}_{Q_YX} = \mathbf{P}_{Q_YX}\mathbf{P}_Y = \mathbf{0}$.

(4) Holds if and only if $\text{Sp}(\mathbf{X})$ and $\text{Sp}(\mathbf{Y})$ are disjoint. The matrices $\mathbf{Q}_X$ and $\mathbf{Q}_Y$ in $\mathbf{P}_{X/Q_Y}$ and $\mathbf{P}_{Y/Q_X}$ are called metric matrices. Note that the two terms in this decomposition are not mutually orthogonal. However, define a metric matrix,

$$\mathbf{K}^* = \mathbf{Q}_X + \mathbf{Q}_Y + \mathbf{TDT}',$$

where $\mathbf{T}$ is such that $\text{Sp}(\mathbf{T}) = \text{Ker}(\mathbf{Z}')$ and $\mathbf{D}$ is an arbitrary positive definite matrix. Then, $\mathbf{P}_Z = \mathbf{P}_{Z/K^*}$, $\mathbf{P}_{X/Q_Y} = \mathbf{P}_{X/K^*}$, $\mathbf{P}_{Y/Q_X} = \mathbf{P}_{Y/K^*}$, and $(\mathbf{P}_{X/K^*})'\mathbf{K}^*\mathbf{P}_{Y/K^*} = \mathbf{0}$. (That is, $\mathbf{P}_{X/Q_Y} = \mathbf{P}_{X/K^*}$ and $\mathbf{P}_{Y/Q_X} = \mathbf{P}_{Y/K^*}$ are mutually orthogonal under the metric implied by $\mathbf{K}^*$.) The matrix $\mathbf{K}^*$ is an example of orthogonalizing metric defined by Rao and Mitra ([5], Lemma 5.3.1).

(5) Arises when we impose a constraint of the form $\mathbf{B}'\mathbf{b} = \mathbf{0}$ on the vector of regression coefficients, $\mathbf{b}$. Note that this constraint can equivalently be written as $\mathbf{b} = \mathbf{A}\tilde{\mathbf{b}}$, where $\tilde{\mathbf{b}}$ is the reduced (reparameterized) coefficient vector. Note that $\mathbf{P}_{ZA}\mathbf{P}_{Z(Z'Z)^-Z'B} = \mathbf{P}_{Z(Z'Z)^-Z'B}\mathbf{P}_{ZA} = \mathbf{0}$.

We can readily extend the above decompositions into K-orthogonal cases where $\mathbf{K}$ is a *nnd* metric matrix such that $\text{rank}(\mathbf{KX}) = \text{rank}(\mathbf{X})$ and $\text{rank}(\mathbf{KY}) = \text{rank}(\mathbf{Y})$. Define $\mathbf{P}_{Z/K} = \mathbf{Z}(\mathbf{Z}'\mathbf{KZ})^-\mathbf{Z}'\mathbf{K}$ and $\mathbf{Q}_{Z/K} = \mathbf{I} - \mathbf{P}_{Z/K}$. Then, under the rank conditions stated above $\mathbf{P}^2_{Z/K} = \mathbf{P}_{Z/K}$, $\mathbf{Q}^2_{Z/K} = \mathbf{Q}_{Z/K}$, $(\mathbf{KP}_{Z/K})' = \mathbf{KP}_{Z/K}$, $(\mathbf{KQ}_{Z/K})' = \mathbf{KQ}_{Z/K}$, $\mathbf{P}_{Z/K}\mathbf{Q}_{Z/K} = \mathbf{Q}_{Z/K}\mathbf{P}_{Z/K} = \mathbf{0}$, and $(\mathbf{P}_{Z/K})'\mathbf{KQ}_{Z/K} = (\mathbf{Q}_{Z/K})'\mathbf{KP}_{Z/K} = \mathbf{0}$. We assume that $\mathbf{Z} = [\mathbf{X} \mid \mathbf{Y}]$, and $\text{rank}(\mathbf{KX}) = \text{rank}(\mathbf{X})$ and $\text{rank}(\mathbf{KY}) = \text{rank}(\mathbf{Y})$.

Lemma 4.

1′. $\mathbf{P}_{Z/K} = \mathbf{P}_{X/K} + \mathbf{P}_{Y/K}$ *if and only if* $\mathbf{X}'\mathbf{KY} = \mathbf{0}$.

2′. $\mathbf{P}_{Z/K} = \mathbf{P}_{X/K} + \mathbf{P}_{Y/K} - \mathbf{P}_{X/K}\mathbf{P}_{Y/K}$ *if and only if* $\mathbf{P}_{X/K}\mathbf{P}_{Y/K} = \mathbf{P}_{Y/K}\mathbf{P}_{X/K}$.

3′. $\mathbf{P}_{Z/K} = \mathbf{P}_{X/K} + \mathbf{P}_{Q_{X/K}Y/K} = \mathbf{P}_{Y/K} + \mathbf{P}_{Q_{Y/K}X/K}$.

4′. $\mathbf{P}_{Z/K} = \mathbf{P}_{X/KQ_{Y/K}} + \mathbf{P}_{Y/KQ_{X/K}}$ *if and only if* $\text{Sp}(\mathbf{X}) \cap \text{Sp}(\mathbf{Y}) = \{\mathbf{0}\}$.

5′. $\mathbf{P}_{Z/K} = \mathbf{P}_{ZA/K} + \mathbf{P}_{Z(Z'KZ)^-B/K}$, *where* $\text{Sp}(\mathbf{A}) = \text{Ker}(\mathbf{B}')$ *and* $\mathbf{Z}'\mathbf{KW} = \mathbf{B}$ *for some* $\mathbf{W}$.

Note 19. Note that $\mathbf{P}_{Z/K}$ is a special case of $\mathbf{P}_{Z:L}$, where $\mathbf{L} = \mathbf{KZ}$, and that it reduces to $\mathbf{P}_Z$ when $\mathbf{K} = \mathbf{I}$. Note also that the two terms in decomposition (4′) are not K-orthogonal. However, as in (4), we can find an orthogonalizing metric under which the two terms are K^*-orthogonal. Such a metric is found by $\mathbf{K}^* = \mathbf{KQ}_{X/K} + \mathbf{KQ}_{Y/K} + \mathbf{TDT}'$, where $\text{Sp}(\mathbf{T}) = \text{Ker}(\mathbf{Z}')$, and $\mathbf{D}$ is an arbitrary *pd* matrix.

We now derive decompositions of $\mathbf{P}_{Z:L}$ analogous to (1)–(5) and (1′)–(5′) above:

Theorem 4.

1″. $\mathbf{P}_{Z:L} = \mathbf{P}_{X:M} + \mathbf{P}_{Y:N}$ *if and only if* $\mathbf{M}'\mathbf{Y} = \mathbf{0}$ *and* $\mathbf{N}'\mathbf{X} = \mathbf{0}$.

2″. $\mathbf{P}_{Z:L} = \mathbf{P}_{X:M} + \mathbf{P}_{Y:N} - \mathbf{P}_{X:M}\mathbf{P}_{Y:N}$ *if and only if* $\mathbf{P}_{X:M}\mathbf{P}_{Y:N} = \mathbf{P}_{Y:N}\mathbf{P}_{X:M}$.

3″. $\mathbf{P}_{Z:L} = \mathbf{P}_{X:M} + \mathbf{P}_{Q_{X,M}Y:Q_{M,X}N} = \mathbf{P}_{Y:N} + \mathbf{P}_{Q_{Y:N}X:Q_{N:Y}M}$.

4″. $\mathbf{P}_{Z:L} = \mathbf{P}_{X:Q_{N:Y}M} + \mathbf{P}_{Y:Q_{M,X}N}$ if $|\mathbf{I} - \mathbf{P}_{X:M}\mathbf{P}_{Y:N}| \neq 0$.

5″. $\mathbf{P}_{Z:L} = \mathbf{P}_{ZA:LC} + \mathbf{P}_{Z(L'Z)^{-}B:L(Z'L)^{-}D}$, *where* $\mathrm{Sp}(\mathbf{A}) = \mathrm{Ker}(\mathbf{D}')$, $\mathrm{Sp}(\mathbf{C}) = \mathrm{Ker}(\mathbf{B}')$, $\mathbf{L}'\mathbf{V} = \mathbf{B}$ *for some* $\mathbf{V}$ *and* $\mathbf{Z}'\mathbf{W} = \mathbf{D}$ *for some* $\mathbf{W}$.

Proof. (1″) is nothing but Theorem 5.1.2 of Rao and Mitra [5]. (Note that $\mathbf{P}_{X:M}\mathbf{P}_{Y:N} = \mathbf{P}_{Y:N}\mathbf{P}_{X:M} = \mathbf{0}$ is also an equivalent condition.) (1″) also subsumes (4′) of Lemma 3. (Simply set $\mathbf{M}' = \mathbf{X}'\mathbf{K}\mathbf{Q}_{Y/K}$ and $\mathbf{N}' = \mathbf{Y}'\mathbf{K}\mathbf{Q}_{X/K}$, and verify that $\mathbf{M}'\mathbf{Y} = \mathbf{0}$ and $\mathbf{N}'\mathbf{X} = \mathbf{0}$.)

To prove (2″) we first let $\mathbf{U}$ and $\mathbf{W}$ be two matrices such that $\mathrm{Sp}(\mathbf{U}) = \mathrm{Sp}(\mathbf{X}) \cap \mathrm{Sp}(\mathbf{Y})$ and $\mathrm{Sp}(\mathbf{W}) = \mathrm{Ker}(\mathbf{M}') + \mathrm{Ker}(\mathbf{N}')$. By Theorem 5.1.4 of Rao and Mitra [5] concerning the product of two projectors, the commutativity of $\mathbf{P}_{X:M}$ and $\mathbf{P}_{Y:N}$ implies $\mathbf{P}_{X:M}\mathbf{P}_{Y:N} = \mathbf{P}_{Y:N}\mathbf{P}_{X:M} = \mathbf{P}_{U:W}$. Let $\mathbf{P}_1 = \mathbf{P}_{Z:L} - \mathbf{P}_{X:M}$ and $\mathbf{P}_2 = \mathbf{P}_{Z:L} - \mathbf{P}_{Y:N}$. From Theorem 5.1.3 of Rao and Mitra [5] concerning the difference between two projectors, $\mathbf{P}_1 = \mathbf{P}_{\mathrm{Sp}(Z)\cap\mathrm{Ker}(M'),\mathrm{Sp}(X)\oplus\mathrm{Ker}(Z')}$ and $\mathbf{P}_2 = \mathbf{P}_{\mathrm{Sp}(Z)\cap\mathrm{Ker}(N'),\mathrm{Sp}(Y)\oplus\mathrm{Ker}(Z')}$. Because of the commutativity of $\mathbf{P}_{X:M}$ and $\mathbf{P}_{Y:N}$, $\mathbf{P}_1$ and $\mathbf{P}_2$ are also commutative; i.e., $\mathbf{P} = \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$. Again by Theorem 5.1.4 of Rao and Mitra [5], $\mathbf{P}$ is the projector onto $(\mathrm{Sp}(\mathbf{Z}) \cap \mathrm{Ker}(\mathbf{M}')) \cap (\mathrm{Sp}(\mathbf{Z}) \cap \mathrm{Ker}(\mathbf{N}'))$ along $(\mathrm{Sp}(\mathbf{X}) \oplus \mathrm{Ker}(\mathbf{Z}')) + (\mathrm{Sp}(\mathbf{Y}) \oplus (\mathbf{Z}')) = E^n$, which implies $\mathbf{P} = \mathbf{0}$. The converse is trivial.

(2″) also follows from statement (vi) of Condition (13). We have $\mathbf{I} - \mathbf{Q}_1\mathbf{Q}_2 = \mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_{V_1+V_2,W_1\cap W_2}$, which is nothing but (2″) of Theorem 4 by setting $V_1 = \mathrm{Sp}(\mathbf{X})$, $V_2 = \mathrm{Sp}(\mathbf{Y})$, $W_1 = \mathrm{Ker}(\mathbf{M}')$, and $W_2 = \mathrm{Ker}(\mathbf{N}')$, so that $\mathrm{Sp}(\mathbf{Z}) = V_1 + V_2$ and $\mathrm{Ker}(\mathbf{L}') = W_1 \cap W_2$.

To prove the first equality in (3″), we need to show that $\mathrm{Sp}(\mathbf{Z}) = \mathrm{Sp}([\mathbf{X} \mid \mathbf{Q}_{X:M}\mathbf{Y}])$ and $\mathrm{Sp}(\mathbf{L}) = \mathrm{Sp}([\mathbf{M} \mid \mathbf{Q}_{M:X}\mathbf{N}])$. To prove the former, we simply note that

$$[\mathbf{X} \mid \mathbf{Q}_{X:M}\mathbf{Y}] = \mathbf{Z}\begin{bmatrix} \mathbf{I} & -(\mathbf{M}'\mathbf{X})^{-}\mathbf{M}'\mathbf{Y} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

where the second matrix on the right is nonsingular. $\mathrm{Sp}(\mathbf{L}) = \mathrm{Sp}([\mathbf{M} \mid \mathbf{Q}_{M:X}\mathbf{N}])$ can also be proved similarly. We immediately see $\mathbf{P}_{Z:L} = \mathbf{P}_{[X|Q_{X:M}Y]:[M|Q_{M:X}N]}$, and $\mathbf{X}'\mathbf{Q}_{M:X}\mathbf{N} = \mathbf{0}$ and $\mathbf{M}'\mathbf{Q}_{X:M}\mathbf{Y} = \mathbf{0}$. So if $\mathrm{Sp}(\mathbf{Q}_{X:M}\mathbf{Y}) \oplus \mathrm{Ker}(\mathbf{N}'\mathbf{Q}_{X:M}) = E^n$, this case reduces to (1″). To show $\mathrm{Sp}(\mathbf{Q}_{X:M}\mathbf{Y}) \oplus \mathrm{Ker}(\mathbf{N}'\mathbf{Q}_{X:M}) = E^n$, we first note $\mathrm{Sp}(\mathbf{X})$ and $\mathrm{Sp}(\mathbf{Q}_{X:M}\mathbf{Y})$ are disjoint, and $\mathrm{Sp}(\mathbf{M})$ and $\mathrm{Sp}(\mathbf{Q}_{M:X}\mathbf{N})$ are also disjoint. This implies $\mathrm{rank}(\mathbf{Q}_{X:M}\mathbf{Y}) = \mathrm{rank}(\mathbf{Q}_{M:X}\mathbf{N})$, since $\mathrm{Sp}(\mathbf{Z}) = \mathrm{Sp}([\mathbf{X} \mid \mathbf{Q}_{X:M}\mathbf{Y}])$ and

$\mathrm{Sp}(\mathbf{L}) = \mathrm{Sp}([\mathbf{M} \mid \mathbf{Q}_{X:M}\mathbf{Y}])$. We also note $\mathrm{Sp}(\mathbf{Q}_{X:M}\mathbf{Y})$ and $\mathrm{Sp}(\mathbf{Q}_{M:X}\mathbf{N})$ are disjoint. This, together with $\mathrm{rank}(\mathbf{Q}_{X:M}\mathbf{Y}) = \mathrm{rank}(\mathbf{Q}_{M:X}\mathbf{N})$, implies $\mathrm{Sp}(\mathbf{Q}_{X:M}\mathbf{Y}) \oplus \mathrm{Ker}(\mathbf{N}'\mathbf{Q}_{X:M}) = E^n$. The second equality in (3″) can be proved similarly.

To prove (4″), we note that $\mathrm{Sp}([\mathbf{M} \mid \mathbf{N}]) = \mathrm{Sp}([\mathbf{Q}_{N:Y}\mathbf{M} \mid \mathbf{Q}_{M:X}\mathbf{N}])$, since

$$[\mathbf{Q}_{N:Y}\mathbf{M} \mid \mathbf{Q}_{M:X}\mathbf{N}] = \mathbf{L}\begin{bmatrix} \mathbf{I} & -(\mathbf{X}'\mathbf{M})^{-}\mathbf{X}'\mathbf{N} \\ -(\mathbf{Y}'\mathbf{N})^{-}\mathbf{Y}'\mathbf{M} & \mathbf{I} \end{bmatrix},$$

where the second matrix on the right is nonsingular, if $|\mathbf{I} - \mathbf{P}_{X:M}\mathbf{P}_{Y:N}| \neq 0$. We have $\mathbf{X}'\mathbf{Q}_{M:X}\mathbf{N} = \mathbf{0}$, and $\mathbf{Y}'\mathbf{Q}_{N:Y}\mathbf{M} = \mathbf{0}$, and by a similar argument as in the proof of (3″) above, $\mathrm{Sp}(\mathbf{X}) \oplus \mathrm{Ker}(\mathbf{M}'\mathbf{Q}'_{N:Y}) = E^n$ and $\mathrm{Sp}(\mathbf{Y}) \oplus \mathrm{Ker}(\mathbf{N}'\mathbf{Q}_{M:X}) = E^n$, so that this case again reduces to (1″). Note that in (4), $\mathrm{Sp}(\mathbf{X}) \cap \mathrm{Sp}(\mathbf{Y}) = \{\mathbf{0}\}$ if and only if $|\mathbf{I} - \mathbf{P}_X\mathbf{P}_Y| = 0$, and in (4′), $\mathrm{Sp}(\mathbf{X}) \cap \mathrm{Sp}(\mathbf{Y}) = \{\mathbf{0}\}$ if and only if $|\mathbf{I} - \mathbf{P}_{X/K}\mathbf{P}_{Y/K}| = 0$. However, no such relationship is claimed in (4″). Note also that (4) and (4′) both represent equivalence relations, but (4″) only one directional implication.

To prove (5″), we note $\mathrm{Sp}(\mathbf{Z}) = \mathrm{Sp}([\mathbf{ZA} \mid \mathbf{Z}(\mathbf{L}'\mathbf{Z})^{-}\mathbf{B}])$ and $\mathrm{Sp}(\mathbf{L}) = \mathrm{Sp}([\mathbf{LC} \mid \mathbf{L}(\mathbf{Z}'\mathbf{L})^{-}\mathbf{D}])$, which can readily by established following a similar line of argument as above. We have $\mathbf{AZ}'\mathbf{L}(\mathbf{Z}'\mathbf{L})^{-}\mathbf{D} = \mathbf{0}$ and $\mathbf{C}'\mathbf{L}'\mathbf{Z}(\mathbf{L}'\mathbf{Z})^{-}\mathbf{B} = \mathbf{0}$, and by a similar argument as in the proof of (3″) above, $\mathrm{Sp}(\mathbf{ZA}) \oplus \mathrm{Ker}(\mathbf{C}'\mathbf{L}') = E^n$ and $\mathrm{Sp}(\mathbf{Z}(\mathbf{L}'\mathbf{Z})^{-}\mathbf{B}) \oplus \mathrm{Ker}(\mathbf{D}'(\mathbf{L}'\mathbf{Z})^{-}) = E^n$, so that this case also reduces to (1″). □

Note 20. Note that $\mathrm{Sp}(\mathbf{Z}) \cap \mathrm{Ker}(\mathbf{L}') = \{\mathbf{0}\}$ implies $\mathrm{rank}(\mathbf{Z}) = \mathrm{rank}(\mathbf{P}_L) = \mathrm{rank}(\mathbf{Z}'\mathbf{P}_L\mathbf{Z})$. This, in turn, implies $\mathbf{P}_{Z:L}$ can be rewritten as $\mathbf{P}_{Z/P_L}$. Since under similar conditions $\mathbf{P}_{X:M} = \mathbf{P}_{X/P_M}$ and $\mathbf{P}_{Y:N} = \mathbf{P}_{Y/P_N}$, (1″) can be rewritten as

$$\mathbf{P}_{Z/P_L} = \mathbf{P}_{X/P_M} + \mathbf{P}_{Y/P_N}.$$

This looks similar to decomposition (1′). However, there is a fundamental difference between the two. In the above decomposition metric matrices used to define three projectors are all distinct. We can find a common metric matrix that preserves the above relationship in a manner similar to that in (4). Define $\mathbf{K}^* = \mathbf{P}_M + \mathbf{P}_N + \mathbf{TDT}'$, where $\mathbf{T}$ is such that $\mathrm{Sp}(\mathbf{T}) = \mathrm{Ker}(\mathbf{Z}')$ and $\mathbf{D}$ is an arbitrary *pd* matrix. We then have $\mathbf{P}_{Z/K^*} = \mathbf{P}_{Z/P_L}$, $\mathbf{P}_{X/K^*} = \mathbf{P}_{X/P_M}$, $\mathbf{P}_{Y/K^*} = \mathbf{P}_{Y/P_N}$, and $(\mathbf{P}_{X/K^*})'\mathbf{K}\mathbf{P}_{Y/K^*} = \mathbf{0}$. A proof is straightforward.

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